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THE SCIENCE AND ART

OF

ARITHMETIC.

By A. SONNENSCHEIN and H. A. NESBITT, M.A.

Part I. Integral. Parts II. & III. Fractional & Approximate Calculations.

OPINIONS OF THE PRESS.

"Mr. Sonnenschein is a pupil, and a thoroughly taught pupil, of Mr. De Morgan's, and it is scarcely necessary to say more in order to convince all who know Mr. De Morgan's works that there is nothing like half-digested work in this arithmetic. This first part of Mr. Sonnenschein's book is admirable of its kind, and better fitted for ordinary school use than Mr. De Morgan's *Arithmetic*, which is more suitable to students and teachers. Brevity and lucidity in the exposition of principle are its main characteristics as a scientific book; and great care in the explanation of simple practical rules for shortening or verifying calculations is its main characteristic in reference to the art of computation. It gives a clear proof of all the rules,—insisting on the exact meaning of the various operations and their interpretation,—and contains a remarkably good chapter on the general properties of numbers, so far as they can be explained to beginners who have only mastered the arithmetic of integers. It is hardly possible to speak too well of this little book, which we have examined very carefully."—*Spectator.*

". . . Nor should we notice this lucid and clever work, except with a word of commendation in our short notices, but for the chapter on the ready decimalization of our weights and measures, which is worth the attention of all calculators. . . . Still, it is an ingenious suggestion to decimalize all the different tables of weights and measures by observing the analogies between the relations of those tables and those of the money table, and so making one rule do for all alike. . . . We should add, as we have noticed a particular chapter in this excellent arithmetic, that the book is throughout good, though some portions of it are better adapted for the use of teachers than for the use of pupils. These portions, however, can easily be omitted by the pupil until he is sufficiently advanced. The thoroughness of the methods of proof is exceedingly praiseworthy."—*Spectator.*

"Forty years have elapsed since the appearance of Prof. De Morgan's 'Elements of Arithmetic,' at a time when perhaps few teachers, as they submitted the rules of the science to their pupils, cared to establish them upon reason and demonstration. The effect of this work was that a rational arithmetic began to be taught generally, and the mere committing of rules to memory took its due subordinate position in the course of instruction. Such a method of treatment will go far to develop and exercise the reasoning powers, and in the case of many pupils, there is hardly any other subject which can so well be made a groundwork for the exercise of the reasoning faculty. The book before us is avowedly drawn up in agreement with the principles of Mr. De Morgan's work, and the aim of the authors is to lead the student 'to the discovery of the several rules by some path such as an original discoverer

might have travelled.' In this first part, which treats of Integral Arithmetic, we consider that they have carried out their principles successfully, and hope they will succeed as well with the remaining two parts, which are to embrace respectively Vulgar Fractions and Approximate Calculations. The rules enunciated are few and tersely given; there is a great store of illustration; elementary difficulties are well stated and honestly grappled with, and cleared up in a way that brings the subject to the level of the capacities of junior students; at the same time advanced as well as young teachers may gather much that is useful from the book. A reader who has carefully gone through the work, can hardly fail to master the early details of the science; if he fail, it will not be the fault of the authors. The subjects treated of are numeration, modes of computation, the so-called first four rules, contracted operations, scales of notation, and properties of numbers. Under this last division we have much valuable matter grouped under the several heads of Divisibility of Numbers, Casting out Nines, Resolution into Prime Factors, Greatest Common Measure, and Least Common Multiple. Throughout and at the end of the work occur numerous examples, very varied, all of which are carefully arranged, and many fully worked out in two or more ways. With this short analysis of the contents, we heartily commend the work to teachers generally, assuming, of course, that they will regulate their use of it in proportion to the requirements of age and ability of their pupils. The work is neatly got up, and we have detected hardly any errata."—*Nature.*

"The authors of this excellent school arithmetic are to be congratulated on having brought their work to a successful termination. . . . The same good arrangement, ample store of illustration, and copious examples for practice, are to be found in this volume as had place in the first. . . . In this volume we have more advanced subjects treated in like manner. But an analysis of the contents will give a good idea of the work. Under Part II. we have the subject of Vulgar Fractions clearly treated, with applications to Practice, and a chapter which treats of Proportion, the Chain Rule, Compound Proportion, and Proportional Parts. In Part III. are chapters on Converging Fractions, Decimals with their properties, and several applications to Money, Weights, Measures, &c., the Metric System, Progressions, Interest, Discount, Stocks, Evolution, and a good chapter on Arithmetical Complements. There is also a chapter in which we have Continued Product to a given limit, Compound Interest, Equation of Payments, Complex Decimals, Duodecimals, and International Calculations. At the end of the work are given 250 Miscellaneous Exercises. There is enough here to satisfy any youthful arithmetician, and the methods employed are the 'latest out.' The complete work gives ample evidence that it is the composition of men who have given much time and thought to the subject, and have had much tuitional experience."—*Nature.*

"Decidedly the clearest, most useful, and easiest method we have seen of teaching the principles and practices of arithmetic. It combines the excellences of Colenso, De Morgan, and Barnard Smith, with excellences peculiar to itself."—*The Rock.*

"It is a very original and well-reasoned system of educating the mind by means of numbers. The authors, working upon the principle 'that the student must be led to the *discovery* of the several rules by some path such as an original discoverer might have travelled,' have really begun at the beginning and logically deduced one step from another, making all so clear as they proceed, that the merest beginner should understand not merely the 'how' but the 'why.' It is perfectly true that more time and space than some may judge necessary are occupied in presenting what is merely one and the same fact under different aspects, and that what appears to be a complete system of arithmetic may be, and often is, packed into less space than the volume before us, which is but the first of three parts. Those, however, who have any experience in teaching, or perhaps remember their own difficulties in

working by rule of thumb, will entirely agree with the authors that progress is not mere advance from rule to rule. Any process once properly realized can never be quite forgotten, and to impart to students what our authors term 'a thorough and all but visual realization of each process,' should be the aim of every teacher. They can scarcely have a more efficient book to work with than that of Messrs. Sonnenschein and Nesbitt."—*Standard.*

"Some little time ago we drew attention to Part I. of Messrs. Sonnenschein and Nesbitt's Science and Art of Arithmetic. We have now Parts II. and III., dealing with vulgar fractions and approximate calculations, in which again we notice an independence of thought and originality of treatment which go far to shew how little the educational value of arithmetic has hitherto been understood."—*Standard.*

"If we mistake not, Messrs. Sonnenschein and Nesbitt's volume will altogether revolutionize the old methods of teaching what has hitherto been supposed a dry study. It is divided into three parts, viz., 'Integral, Vulgar Fractions, and Approximate Calculations.' New methods are given for very rapid, and in most cases, 'mental,' decimalization of money, weights and measures to any assigned degree of accuracy, and for the ready intro-conversion of the coins, weights, and measures of different nations. The volume must supersede the old-fashioned methods, and will be invaluable to young teachers, as well as learners, and will rapidly make its way as 'the' text-book of arithmetic."—*Naval and Military Gazette.*

"This is a work on Arithmetic of a peculiar, and in some respects an original character. Following in the steps of Professor De Morgan, the chief aim of the authors in explaining the *rationale* of the various arithmetical processes is not to give logical demonstrations of the several rules which a student is required to learn, but to carry him along some such path of reasoning as must have been travelled by an original discoverer; the present concise and conventional processes being unravelled, so to speak, and traced up to their first principles. . . . There is much to recommend in this view of teaching arithmetic; for, as the authors remark in the preface, no subject is so well fitted as this for the early training of the reasoning powers, 'principally because the student is enabled, without apparatus of any kind, steadily to test all his *à priori* conclusions by the light of experience.'"—*Educational Times.*

"This is the second part of a treatise on Arithmetic, the first of which has been already favourably noticed in these columns. The subject of Fractions is here taken up and treated with the fulness and completeness due to the important place it occupies as the Key-stone of Arithmetic. . . . In dealing with Decimals the authors have introduced the principle of 'Approximate Calculations,' by means of which much trouble is saved in working out results. . . . It is here that the great utility and simplicity of Decimals is seen, and the book before us does good service in giving prominence to this feature. . . . We are disposed altogether to entertain a high opinion of the merits of this work. The way in which it leads up to the various rules, by mental calculations and other preparatory steps, which serve to break the difficulty felt by the young learner, and its general adherence throughout to the best principles and methods of teaching, distinguish it as a work out of the common run on treatises of School Arithmetic."—*Educational Times.*

"We have no hesitation in saying that the right note has been struck by the authors of the little book, a second edition of which lies before us. . . . We are happy to conclude with an assurance that, take it as it is, the work of Messrs. Sonnenschein and Nesbitt is the best in the language on the subject of which it treats."—*Inquirer.*

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THE SCIENCE AND ART

OF

ARITHMETIC;

For the Use of Schools.

PART I. INTEGRAL.

BY

A. SONNENSCHEIN

AND

H. A. NESBITT, M.A., UNIV. COLL., LONDON.

"The mills of God grind slowly, but they grind exceeding small."

THIRD EDITION, REVISED AND ENLARGED.

FIFTH THOUSAND.

LONDON:

WHITTAKER AND CO.

1876

LONDON:
PRINTED BY C. GREEN AND SON, 178, STRAND.

TO

MISS PIPE,

OF LALEHAM, CLAPHAM PARK,

This Work is inscribed,

IN TOKEN OF THEIR RESPECT AND GRATITUDE,

BY

THE AUTHORS.

PREFACE TO THE FIRST EDITION.

There is no need now to insist on a rational study of Arithmetic. It is admitted on all sides that no subject is so well fitted for the early training of the reasoning powers, and principally because the student is enabled, without apparatus of any kind, steadily to test all his a priori conclusions by the light of experience. In History, Physics, and even in Language, the student must have premisses supplied him; but his Mathematical studies can all be "evolved from his inner consciousness."

Ever since the pernicious plan of teaching by mere rote and rule of thumb was abandoned, the teaching of Elementary Mathematics has steadily risen towards higher levels, and we may perhaps be allowed to note down some of the most remarkable stages. A certain school of teachers very early felt the necessity of enlisting the child's reason on their side; but the means they adopted were not always wise or even honest. In a modest little work on Vulgar Fractions, which is otherwise very meritorious, we find the following "proof" of the formula $\frac{a}{b} = \frac{ma}{mb}$:—$\frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \frac{6}{6} = \frac{7}{7} = \frac{8}{8}$, &c. If of $\frac{2}{2}$ both terms be multiplied by 3, we obtain $\frac{6}{6}$; by 4, $\frac{8}{8}$; and so on. If, on the contrary, of $\frac{10}{10}$, both terms be divided by 5, we obtain, $\frac{2}{2}$, &c.; hence (!), &c. &c.

We can imagine that inexperienced children would readily give their assent to such a proof; but the teacher ought to have known, 1st, that $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, &c., are only integers very thinly disguised as fractions, and that it does not follow that what is true of integers is

necessarily true of fractions; 2nd, that from this proof we might equally deduce $\frac{a}{b} = \frac{a \pm c}{b \pm c}$ which, though true for these disguised units, is false for fractions.

Another school of teachers have resolutely shrunk from such slurring over of difficulties, and are satisfied with nothing short of irrefragable proof. This is, of course, a great step in advance, but they fail in one point. The proof in each case, is *given* to the student, and he has to learn it. Now following and even mastering and remembering a chain of reasoning, though it certainly is an instruction and even a discipline, is not yet true education.

The disciples of Professor De Morgan well know that "each new notion to be acquired must be attached to and assimilated with the notions already existing in the mind." The logical consequence of this maxim is, that the student must be led to the *discovery* of the several rules by some path such as an original discoverer might have travelled. Thus each rule furnishes the *raison d'être* of its successor; for example, Subtraction must lead through Cumulative Subtraction to Division, which, in its turn, leads to Fractions, and these again lead to the idea of Ratio and Proportion. How absurd, then, were those systems which taught Proportion before Fractions; the advanced notion before the earlier one! The steady educing of new and wider notions from old and narrow ones is true education.

Some might object to this method as being slow in shewing results. Even though we must admit the necessity of "payment by results;" yet we hold that for educational purposes the "seeking of the truth is worth more than the finding of it." In other words, Processes are worth more than Results; the highest wages of work is work itself.

But we deny that the method of teaching here advocated is slow and barren of results. Progress is not synonymous with mere advance from rule to rule. A thorough and all but visual realization of each process, gives the young student a feeling of power and perfect mastery, which obviates the necessity of perpetual recapitu-

lation. An experience of upwards of twenty years has shewn us that the more nearly we have at any time approximated to this our ideal, the better the results have been, even when subjected to the severe test of a competitive examination.

In the following pages we have endeavoured to carry out these principles. The attempt is doubtless imperfect enough; but all shortcomings we trust will be treated indulgently, if we have made but one step further in the right direction. The book is intended to guide young teachers; for although the pupil must use the work for the sake of the examples, he is pretty sure not to read the letter-press. We hope, however, that the teacher will find his work, if not lessened, at least rendered easier by it. The sequence of Chapters is not necessarily a sequence of lessons; indeed, we should recommend that two or more Chapters be taken in hand simultaneously.

The First Part treats of Integral Arithmetic purely; the Second Part, of Vulgar Fractions and the notions immediately deduced from them. The Third Part is devoted to Approximate Calculations, and new methods are given for very rapid, and in most cases mental decimalization of money, weights and measures, to any assigned degree of accuracy, and for the ready interconversion of the coins, weights and measures of different nations. We have reason to believe that these methods will prove a considerable boon to the commercial world.

A. SONNENSCHEIN,
H. ARTHUR NESBITT.

JAN. 1870.

PREFACE TO THE SECOND EDITION.

We think it desirable to make to teachers using this book the following suggestions, the results of our own experience:

1. In teaching the series *a—as* (pp. 9, 10), ask questions after each series upon that and all the preceding series, and also give rows of digits for addition similar to that given in § 3, p. 11, adapted to the stage arrived at; e.g. at the stage *u*, ask questions such as the following: 25 + 4? 37 + 6? 28 + 2? 91 + 3? &c.; and write on the board a row of digits for addition containing no numbers higher than 6; e.g. 2, 5, 3, 1, 4, 2, 1, 5, 3, 6. *Ans.* 32.

2. Give much practice in Notation. The Answer-book will furnish numerous examples. Also in Analysis. (Ch. I. p. 7.)

3. We have altered the note on p. 4, to which we invite special attention.

4. Insist upon constant examination and interpretation of results, so as to preclude the possibility of receiving answers manifestly absurd. The answers to Ex. XX. (1) and (2) are 135 *times* and 217 *articles* respectively, and not vaguely 135 and 217, which numbers some pupils are tempted to treat as money. Again, the answer to (16), p. 165, cannot possibly exceed 10*s.* 9¾*d.* per gallon, nor fall short of 7*s.* 6*d.* per gallon; and pupils should, wherever possible, be required to verify results by some rough guess at the answer that might have been expected.

A. SONNENSCHEIN.
H. A. NESBITT.

FEB. 1873.

PREFACE TO THE THIRD EDITION.

SINCE the publication of the Second Edition of Part I., we have published our "A B C of Arithmetic," in which we have given a full analysis of the Multiplication Table. We have consequently omitted from the present edition the portion of this analysis given in former editions.

To accompany the "A B C of Arithmetic" we have also published an Apparatus, described in the note on p. 4, *infra*.

Guided by our experience in using the book, we have made the following additions:

1. In Ex. XIV. two different processes had been treated together; these have been separated by premising Exercises XIV. (*a*) and (*b*), the former Ex. XIV. becoming XIV. (*c*); p. 62.
2. A paragraph on "Continued Product" has been introduced on p. 65.
3. To bridge over the passage from Ex. XIX. to Ex. XX., we have added Ch. VII. § 4 (p. 78), and a corresponding Exercise, viz. Ex. XX. (*a*).
4. Casting out elevens has been transferred from the Preface of the Second Edition to the body of the book, p. 137; and a comparison of the relative values of the tests by casting out nines and elevens is given, p. 159.
5. Criteria for divisibility of numbers has been applied to £. *s.* *d.*, p. 160.

Instead of Ex. XXVI. of the First and Second Editions, we have given an application of the same principle of more general utility, p. 99.

The Table of Contents has been amplified.

A. SONNENSCHEIN,
H. A. NESBITT.

APRIL, 1876.

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ARITHMETIC.

CHAPTER I.

NUMERATION AND NOTATION.

1. THE word Arithmetic is derived from the Greek word *Arithmos*, number, and designates the study of number. It divides itself naturally into two branches, viz. such calculations as can be done without the aid of signs or symbols, and such as require this aid. The former is called Mental Arithmetic; the latter, Slate Arithmetic, or, more properly, Written or Symbolic Arithmetic. It is of this latter branch that we propose chiefly to treat.

2. The word symbol also is Greek, and has the same force as the Latin "sign" or the Germanic "mark;" the figures, 1, 2, 3, &c., being symbols, signs or marks of their respective quantities. Symbols have a language of their own, amenable to all the rules of logic, to the illustration of which rules they are especially adapted; and we are endeavouring, in the study of Arithmetic, to acquire a first knowledge of this peculiar language.

3. When we reckon with counters, say pebbles, each pebble may be made to represent one or more things: if we had, for example, to sum up the number of oxen in two herds, we might count two collections of pebbles, each of which should be equal in number to one of the herds of oxen. There would be no connection between a pebble and an ox further than this, that the pebble is a representative or a *symbol* for the ox. Similarly, written strokes or dots may be made to render a like service, and hence our whole system of Symbolic Arithmetic. [The word *calculation* is derived from the

Latin *calculus*, a pebble, and the professional calculator in ancient Rome, by means of slave labour, adopted some such mechanical method of computation.]

4. The first step necessary to an advance in this study was to agree upon a fixed set of symbols, which should be the alphabet of the common language. Several such attempts have been made; the two sets of symbols in present use are the Roman and the Arabic. The principal Roman symbols are I, V, X, L, C, D, M, signifying respectively one, five, ten, fifty, one hundred, five hundred, and one thousand.*

EXERCISE A.

Read from the board the following:

XII, XXV, XXXVIII, L, LX, LXV, LXXIII, LXXXVII, C, CC, CCL, CCCLX, CCCCIII, D, DL, DCLXXXVI, M, MC, MD, MDC, MDCLXXXVIII, MDCCLX, MDCCCXV, MDCCCLXVII.

5. The Romans made a remarkable addition to their scheme of notation, by making the function of the symbol occasionally depend on its position: thus, VI denotes five *and* one, or six; but IV denotes five *diminished by* one, or four. If, then, the smaller quantity precede the larger, it is to be taken away from the larger; if it follow, to be added; consequently, XL denotes forty, and LX sixty.

EXERCISE B.

Read from the board:

XL, LX, XC, CX, MDXCIV, MDCCLXXXIX, MDCCCXLVIII, MDCCCLXIV.

6. These symbols are of little use for complicated calculations, for many reasons, but chiefly for this; that their value depends on their shape, and not at all on their position. The set of symbols in common use relies for its power of expressing any number, however large, almost exclusively on the artifice that the value of each figure increases tenfold for every place it is moved further to the left, and this has entailed the necessity of introducing a symbol for *nothing*,

* For an account of the origin of these and other numerical symbols, see Penny Cyclop., art. Numeral Characters.

viz. 0,* and so important is this symbol, that from the word cipher, the whole art has derived its name, Ciphering. These symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and they are called Arabic Figures, because they came to us from the Moors of Spain, the most cultivated people of the Middle Ages. They, however, were not the inventors, having received them from the Hindoos,† among whom they were in use about the commencement of the Christian era.

7. We may write down any number with only two symbols, viz. 0 and 1, if we agree that the value of the symbol 1 shall be doubled every time we move it one step further to the left. Thus,

one	=	1,		
two	=	10,		
three	=	11,	being	one two and one.
four	=	100,	„	a double two.
five	=	101,	„	a double two and one.
six	=	110,	„	a double two and a two.
seven	=	111,	„	a double two, a two, and one.
eight	=	1000,	„	the four doubled, &c.

EXERCISE C.

Write down all the numbers up to 65 on this system of notation, which is called the Binary Scale, from the Latin *bini*, two at a time.

8. Similarly we may write down any number with three symbols, viz. 0, 1, 2, if we agree that the value of each symbol shall be trebled every time we move it one step further to the left. Thus,—

one	= 1		six	= 20	
two	= 2		seven	= 21	
three	= 10		eight	= 22	
four	= 11		nine	= 100	
five	= 12			&c.	

* Read *naught*, which means *nothing*, and not *aught*, which means *anything* (cf. "For aught I know").

† "Dutch" clocks only *come* from Holland, but are *made* in the Black Forest; similarly "Hamburg" grapes *come* from Hamburg, but do not *grow* there.

9. Or we may employ the four symbols 0, 1, 2, 3, if we agree that the value of each symbol shall be quadrupled every time we move it one step further to the left.

10. The Scale of Notation requiring three symbols (0, 1, 2) is called the Ternary Scale; that with four symbols (0, 1, 2, 3) the Quaternary Scale. Similarly we have the Quinary, Senary, Septenary, Octonary, Nonary, Denary or Decimal, Undecimal, Duodecimal Scales, &c., requiring respectively 5, 6, 7, 8, 9, 10, 11, and 12 symbols. The Undecimal and Duodecimal Scales would of course require new symbols, viz. for ten in the Undecimal, and for ten and eleven in the Duodecimal Scale. The most useful of all these is the Decimal Scale, because, owing to our having ten fingers, we count by tens; thus eleven means one and ten; twelve means two and ten; thirteen means three and ten; fourteen, four and ten, &c.; twenty means twain tens; twenty-one, two tens and one, &c.

Exercise D.

On the Ternary Scale write all the numbers up to 82; on the Quaternary, up to 65; on the Quinary, up to 126; on the Senary, to 37; on the Duodecimal (making *t* stand for ten and *e* for eleven), up to 145.

Note. The teacher will find it advisable to deal with concrete rather than with abstract numbers. The authors have published an apparatus consisting of: (*a*) a small case opened by an upper and lower lid; the upper compartment contains one hundred cubes, a centimetre each way; the lower, twenty "staves," each ten times the size of a cube, and one "plate," ten times the size of a stave, or a hundred times the size of a cube: (*b*) a box containing ten such plates, equivalent to a hundred staves, or a thousand cubes: (*c*) a long box containing frameworks to illustrate ten thousand, a hundred thousand, and a million. This apparatus will also be found useful at many stages of the whole course of Arithmetic.—See the A B C of Arithmetic, Part II., Appendix.

11. On Decimal Notation.

(*a*) Write down all the numbers up to one hundred.

(*b*) Read off 43.

Question. Why does this represent 43 and not 7?

*Answer.** Because in the Decimal Scale the value of the symbol is increased tenfold by moving the symbol one place to the left, so the 4 represents four tens, the 3 three units; [or on the apparatus, 43 stands for four staves and three cubes.]

Q. If you had to arrange 43 marbles in heaps of ten each, how many such heaps would there be?

A. Four heaps, and three marbles over.

Q. Would not the three marbles give another heap?

A. No; I am seven marbles short.

Q. If these seven marbles were added, how many heaps of ten would there then be?

A. Five heaps.

Q. Write that down.

A. 50. (And so on with other numbers under one hundred.)

Q. Read off 99.

A. Ninety-nine.

Q. Why ninety-nine, and not eighteen?

A. Because it means nine tens and nine ones.

Q. If you had to draw ninety-nine pounds from the Bank, how many ten-pound notes should you receive?

A. Nine ten-pound notes.

Q. And nothing more?

A. Yes, and nine sovereigns in gold.

Q. Would not these make up another ten-pound note?

A. No; I should be one pound short.

Q. If that pound were supplied, how many ten-pound notes would you then have?

A. Ten ten-pound notes.†

Q. Write that down.

A. £100.

* These answers will not be obtained fully at first. The questions must be so amplified as to lead to them, the requisite amount of amplification depending on the ability of the class; but the subject should not be carried further until each step is thoroughly mastered.

† Never allow the answer "ten," but insist on a reply to the question, "ten what?" *Ans.* Ten pounds, ten times, &c., as the case may be.

Q. How many tens are there in one hundred?

A. Ten tens.

Q. How many units are there in one hundred?

A. One hundred units.

Q. Read off 173.

A. One hundred and seventy-three.

Q. Why should this mean one hundred and seventy-three, and not eleven?

A. Because it means one hundred, seven tens, and three ones.

Q. If 173 marbles were arranged in heaps of ten marbles each, how many such heaps would there be?

A. Seventeen heaps and three marbles over.

Q. Prove it.

A. Because one hundred gives ten tens, and seven more tens are seventeen tens.

[*Q.* Represent this quantity on the apparatus.

A. One plate, seven staves, and three cubes.

Q. Would not these three units give another ten?

A. No; there are seven units short.

Q. If these seven units were added, how many tens would you then have?

A. Eighteen tens.

Q. How many hundreds can be obtained from 173 units?

A. One hundred, and seventy-three units over.

Q. Would not the 73 units yield another hundred.

A. No; I am twenty-seven units short.

Q. If these 27 units were supplied, how many hundreds would you then have?

A. Two hundreds. (And so on with other numbers under one thousand.)

Q. Read off 999.

A. Nine hundred and ninety-nine.

Q. Why should this mean nine hundred and ninety-nine, and not twenty-seven?

A. Because it represents nine hundreds, nine tens and nine units.

Q. If 999 marbles were arranged in heaps of ten, how many such heaps would there be?

A. Ninety-nine heaps of ten, and nine marbles over.

Q. Prove it.

A. Nine hundreds yield ninety tens, and nine more tens make ninety-nine tens.

Q. Would not the nine units which are over give another ten?

A. No; I am one unit short.

Q. If that unit were supplied, how many tens would you then have?

A. One hundred tens.

Q. How many hundreds could you get out of 999 marbles?

A. Nine hundreds, and ninety-nine marbles over.

Q. Would not these ninety-nine marbles yield you another hundred?

A. No; I am one marble short.

Q. If that marble were supplied, how many hundreds would you then have?

A. Ten hundreds.

Q. Write it down.

A. 1000.

Q. Read off 5743. (And so on with numbers under 10000, as was done with the numbers 43 and 173.)

After some practice the pupils can be trained to answer as follows:

Q. Read off 5872.

A. Five thousand, eight hundred and seventy-two units; or five hundred and eighty-seven tens, and two units over, wanting eight units to make up 588 tens; or fifty-eight hundreds, and seventy-two units over, wanting 28 units to make up fifty-nine hundreds; or five thousands, and eight hundred and seventy-two units over, wanting one hundred and twenty-eight units to make up six thousands.

Next, treat 9999 as 99 and 999 have been treated, and work as before till all numbers under 100000 are rendered familiar. Next, treat 99999 and numbers up to 1000000 similarly. Beyond this it will not be found necessary to go, because higher numbers are very rarely required, and if they are present no new difficulty.

12. The following table of symbolic headings to the several columns should be learnt by heart gradually, as the necessity arises, during the progress of the study of section 11.

$$
\begin{array}{llllll}
X & = X.I & & & & \\
C & = X.X & = C.I & & & \\
M & = X.C & = C.X & = M.I & & \\
XM & = X.M & = C.C & = M.X & = XM.I & \\
CM & = X.XM & = C.M & = M.C & = XM.X & = CM.I \\
MM \text{ or } M^2 & = X.CM & = C.XM & = M.M & = XM.C & = CM.X = M^2.I
\end{array}
$$

NOTE. This table is read :

Ten equals ten times one.

One hundred equals ten times ten, equals a hundred times one.

One thousand equals ten times a hundred, or a hundred times ten, or a thousand times one.

One ten-thousand (a myriad) equals ten times a thousand, or a hundred times a hundred, or a thousand times ten, or ten thousand times one.

One hundred-thousand (a lac) equals ten times a ten-thousand, or a hundred times a thousand, or a thousand times a hundred, or ten thousand times ten, or a hundred thousand times one.

One million equals ten times a hundred-thousand, or a hundred times a ten-thousand, or *a thousand times a thousand*, or ten thousand times a hundred, or a hundred thousand times ten, or a million times one.

[Point out that just as a *balloon* is a big ball, a saloon a big *salle*, so a million is a big mille, viz. a thousand thousand, and that this nomenclature is analogous to the word gross, a big dozen, viz. a dozen dozen.]

13. Hitherto we have treated Numeration analytically; we proceed to the synthesis.

Learn by heart: *Ten is a one-cipher number occupying the second place.*

Write down forty, seventy, ninety, sixty-eight, fifteen, thirty-nine, &c.

A hundred is a two-cipher number occupying the third place.

Write down one hundred, five hundred, seven hundred, eight hundred and forty-three, seven hundred and fifty-nine, six hundred and twenty, six hundred and two, two hundred and six, two hundred and sixteen, six hundred and twelve, six hundred and twenty-one, &c.

A thousand is a three-cipher number occupying the fourth place, a

ten thousand a four-cipher number occupying the fifth place, a hundred thousand a five-cipher number occupying the sixth place, and a million a six-cipher number occupying the seventh place. [The whole subject of Numeration and Notation is more fully treated in the "A B C of Arithmetic," Parts I. and II.]

14. Unity followed by any number of ciphers exceeds by unity the quantity expressed by the same number of nines: thus 1000 exceeds 999 by 1; and since the largest numbers by which each of the places can be filled up is 9, unity in any place is always greater in value than all that is to the right of it.

CHAPTER II.

ON MODES OF COMPUTATION.

1. To read off numbers with fluency, group them into periods of three, making the units' figure the last figure of a period, and remembering that the lowest period of three is read off as units, the next as thousands, the next as millions, and so on; e.g. 378,894,216 will be read off 378 millions, 894 thousands, 216.

The same method of grouping quantities will be found advantageous in writing numbers from dictation.

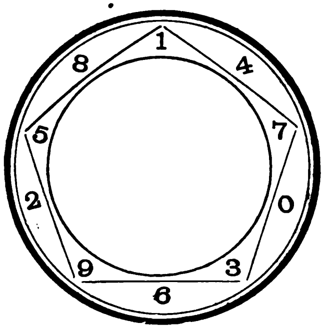
2. Learn to count as follows, *uttering the words as fast as you can talk*:

a.	1, 2, 3, 4, 5, &c. up to		100	*k.*	1, 6, &c. up to	......	101
b.	1, 3, 5, 7	......	101	*l.*	2, 7	......	102
c.	2, 4, 6, 8	......	100	*m.*	3, 8	......	103
d.	1, 4, 7, 10	......	100	*n.*	4, 9	......	104
e.	2, 5, 8, 11	......	101	*o.*	5, 10	......	100
f.	3, 6, 9, 12	......	102	*p.*	1, 7, 13	......	103
g.	1, 5, 9	......	101	*q.*	2, 8, 14	......	104
h.	2, 6, 10	......	102	*r.*	3, 9, 15	......	105
i.	3, 7, 11	......	103	*s.*	4, 10, 16	......	100
j.	4, 8, 12	......	100	*t.*	5, 11, 17	......	101
				u.	6, 12, 18	......	102

v.	1, 8, 15, &c. up to		106
w.	2, 9, 16	......	100
x.	3, 10, 17	......	101
y.	4, 11, 18	......	102
z.	5, 12, 19	......	103
aa.	6, 13, 20	......	104
ab.	7, 14, 21	......	105
ac.	1, 9, 17	......	105
ad.	2, 10, 18	......	106
ae.	3, 11, 19	......	107
af.	4, 12, 20	......	100
ag.	5, 13, 21	......	101
ah.	6, 14, 22, &c. up to		102
ai.	7, 15, 23	......	103
aj.	8, 16, 24	......	104
ak.	1, 10, 19	......	100
al.	2, 11, 20	......	101
am.	3, 12, 21	......	102
an.	4, 13, 22	......	103
ao.	5, 14, 23	......	104
ap.	6, 15, 24	......	105
aq.	7, 16, 25	......	106
ar.	8, 17, 26	......	107
as.	9, 18, 27	......	108

Learn the series *a*, *b*, and *c* backwards.

The following ring will facilitate the learning of the series, for the addition of threes, fours, fives, sixes and sevens, viz. *d—f*, *g—j*, *k—o*, *p—u*, and *v—ab*.



For *d*, *e*, *f*, begin with 1, 2, 3, respectively, and read off the successive figures, travelling *with* the hands of a clock, the ring supplying the figures in the units' place only.

For *g*, *h*, *i*, *j*, begin with 1, 2, 3, 4, respectively, taking every alternate figure, but travel in the direction *opposite* to the hands of a clock.

For *k*, *l*, *m*, *n*, *o*, begin with 1, 2, 3, 4, 5, respectively, and alternate with the figure *diametrically* opposite.

For *p*, *q*, *r*, *s*, *t*, *u*, begin with 1, 2, 3, 4, 5, 6, respectively, taking every alternate figure, and again travelling *with* the hands of a clock.

For *v, w, x, y, z, aa, ab*, begin with 1, 2, 3, 4, 5, 6, 7, respectively; take every figure, but travel in the *opposite* direction to the hands of a clock.

For *ac—aj*, begin with 1, 2, 3, 4, 5, 6, 7, 8, respectively, and remembering that 8 is 10 all but 2, add 10 and deduct 2.

To add nine, *ak—as*, add ten and deduct one.

In the series *ak—as*, note that the ultimate sum of the digits (separate figures) of each number equals the initial number of the series; e.g. 7, 16, 25, 34, 43, 52, 61, 70, 79, 88, 97, 106, &c. 7 = 7, 1 and 6 = 7, 2 and 5 = 7, &c.; 7 and 0 = 7, 7 and 9 = 16, but 1 and 6 again = 7, 8 and 8 = 16, and 1 and 6 = 7, &c.

Or take *ak:* 1, 10, 19, 28, &c. :—1 = 1, 1 and 0 = 1, 1 and 9 = 10, but 1 and 0 = 1, &c.

Or take *as:* 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, &c. Commencing with 9, we have 1 and 8 = 9, 2 and 7 = 9, &c.

3. *Question.* Add the digits of:—7, 9, 3, 6, 2, 4, 7, 8, 5, 1, 2, 9, 3, 7, 1, 6, 8, 5.

Answer. Ninety-three.

Q. Add aloud.

A. "7, 16, 19, 25, 27, 31, 38, 46, 51, 52, 54, 63, 66, 73, 74, 80, 88, 93."

N.B. Never allow "7 and 9 are 16, 16 and 3 are 19, 19 and 6 are 25," and so on, as this wording is fatal to rapidity.

4. Casting out Nines. Add the above series, rejecting nine as fast as it is obtained, and therefore disregarding the figure 9 itself whenever it occurs; thus: 7 (omitting 9) and 3 is 10, rejecting 9 leaves 1, and 6 is 7, and 2 is 9, rejecting this leaves 0; 4 and 7 is 11, rejecting 9 leaves 2, and 8 is 10, rejecting 9 leaves 1, and so on. *Wording:* 7, 10, 1, 7, 9, 0, 4, 11, 2, 10, 1, 6, 7, 9, 0, 3, 10, 1, 2, 8, 16, 7, 12, 3. Now it will be found that the ultimate sum of the digits of the result (in this case 93) will also be 3. That this

coincidence always holds will be proved further on, and it is a useful test of the correctness of the addition. Practise examples on §§ 3 and 4 until very great fluency is obtained. Time spent on this subject will be found a most profitable investment for the teacher, and still more for the pupil, who will after this one effort be relieved of much irksome drudgery in subsequent operations.

5. ON DOZENS. One shilling is worth twelve, or one dozen, pence, therefore every dozen articles will cost as many shillings as each article costs pence; e.g. if 1 article costs 7 pence, a dozen costs 7 shillings; if 3 pence, 3 shillings; if 9 pence, 9 shillings, and so on.

EXERCISE E.

Case I. Find the cost of a dozen things at		2 pence each.	*Ans.*	2*s.*
„	„	4 pence	„	4*s.*
„	„	8 pence	„	8*s.*
„	„	5 pence	„	5*s.*

Case II. Find the cost of a dozen things at 1 shilling and 2 pence each. One shilling is 12 pence, one shilling and 2 pence is 14 pence, so the dozen costs 14*s.*

EXERCISE F.

Find the cost of a dozen	things at	1*s.* 5*d.* each.	*Ans.*	17*s.*
„	„	1*s.* 7*d.*	„	19*s.*
„	„	1*s.* 1*d.*	„	13*s.*
„	„	1*s.* 8*d.*	„	20*s.* = £1.
„	„	1*s.* 11*d.*	„	23*s.* = £1. 3*s.*
„	„	1*s.* 9*d.*	„	21*s.* = £1. 1*s.*

Learn by heart:

(1) 1 shilling = 12 pence.
2 shillings... 24
3 36
4 48
5 60
6 72
7 84
8 96
9 108
10 120
11 132
12 144

(2) 20 shillings = £1.
30 .... £1. 10*s.*
40 .... £2.
50 .... £2. 10*s.*
60 .... £3.
70 .... £3. 10*s.*
80 .... £4.
90 .... £4. 10*s.*
100 .... £5.

Case III. Find the cost of a dozen things at 6*s.* each. 6*s.* is 72 pence, a dozen will cost 72*s.*; 70*s.* is £3. 10*s.*, and 72*s.* is £3. 12*s.*

EXERCISE G.

Find the cost of a dozen things at	5*s.* each.	*Ans.*	£3.
„ „	8*s.*	„	£4. 16*s.*
„ „	3*s.*	„	£1. 16*s.*
„ „	4*s.*	„	£2. 8*s.*
„ „	6*s.* 4*d.*	„	£3. 16*s.*
„ „	7*s.* 3*d.*	„	£4. 7*s.*
„ „	4*s.* 11*d.*	„	£2. 19*s.*
„ „	5*s.* 10*d.*	„	£3. 10*s.*

Case IV. Find the cost of a dozen things at 10*s.* each. 10*s.* = 120 pence, the dozen will cost 120*s.*; 100*s.* is £5; 120*s.* is £6. Similarly a dozen things at 11*s.* 4*d.* will cost £6. 16*s.*, since 11*s.* = 132 pence, 11*s.* 4*d.* = 136 pence; 136*s.* = 100*s.* and 36*s.* = £5 and £1. 16*s.* = £6. 16*s.*

EXERCISE H.

Find the cost of a dozen things at	9*s.* 2*d.* each.	*Ans.*	£5. 10*s.*
„ „	10*s.* 3*d.*	„	£6. 3*s.*
„ „	8*s.* 11*d.*	„	£5. 7*s.*
„ „	11*s.* 10*d.*	„	£7. 2*s.*
„ „	12*s.* 6*d.*	„	£7. 10*s.*
„ „	12*s.* 11*d.*	„	£7. 15*s.*

Case V.* If one thing costs half a penny (a half-penny), a dozen things will cost half a shilling, or 6*d.*

Find the cost of a dozen things at $3\frac{1}{2}d.$ each. *Ans.* 3*s.* 6*d.*, because the 3*d.* yields 3*s.* and the $\frac{1}{2}d.$ 6*d.*

If one thing costs one quarter penny (a farthing), a dozen will cost one quarter shilling, or 3*d.*

Find the cost of a dozen things at $5\frac{1}{4}d.$ each. *Ans.* 5*s.* 3*d.*

If one thing costs three quarter pennies (three farthings), a dozen will cost three quarter shillings, or three three-penny pieces, or 9*d.*

Find the cost of a dozen things at $8\frac{3}{4}d.$ each. *Ans.* 8*s.* 9*d.*

* Any difficulty in appreciating this reasoning will be removed by exhibiting to the pupil a shilling and four three-penny pieces side by side with a penny and four farthings.

Exercise K.

Find the cost of a dozen things at	$2\frac{1}{2}d.$ each.	*Ans.*	$2s.$	$6d.$
„ „	$5\frac{1}{4}d.$	„	$5s.$	$3d.$
„ „	$7\frac{3}{4}d.$	„	$7s.$	$9d.$
„ „	$11\frac{1}{4}d.$	„	$11s.$	$3d.$
„ „	$10\frac{1}{2}d.$	„	$10s.$	$6d.$
„ „	$4\frac{1}{2}d.$	„	$4s.$	$6d.$
„ „	$8\frac{1}{4}d.$	„	$8s.$	$3d.$

Find the cost of a dozen things at $2s.\ 3\frac{1}{2}d.$ each. $2s. = 24d.$; $2s.\ 3d. = 27d.$; $2s.\ 3\frac{1}{2}d. = 27\frac{1}{2}d.$, therefore a dozen will cost $27\frac{1}{2}$ shillings $= £1.\ 7s.\ 6d.$

Find the cost of a dozen things at $3s.\ 8\frac{1}{4}d.$ each. $3s. = 36d.$; $3s.\ 8d. = 44d.$; $3s.\ 8\frac{1}{4}d. = 44\frac{1}{4}d.$, therefore a dozen things will cost $44\frac{1}{4}$ shillings $= £2.\ 4s.\ 3d.$

Find the cost of a dozen things at $7s.\ 1\frac{3}{4}d.$ each. $7s. = 84d.$; $7s.\ 1d. = 85d.$; $7s.\ 1\frac{3}{4}d. = 85\frac{3}{4}d.$, therefore a dozen things will cost $85\frac{3}{4}$ shillings $= £4.\ 5s.\ 9d.$

Exercise L.

	s.	d.			£.	s.	d.
Find the cost of a dozen things at	2	$7\frac{3}{4}$	each.	*Ans.*	1	11	9
„ „	8	$5\frac{1}{2}$		„	5	1	6
„ „	9	$10\frac{1}{4}$		„	5	18	3
„ „	3	$2\frac{3}{4}$		„	1	18	9
„ „	1	$11\frac{1}{2}$		„	1	3	6
„ „	2	$7\frac{1}{2}$		„	1	11	6
„ „	3	$6\frac{3}{4}$		„	2	2	9
„ „	5	$8\frac{1}{2}$		„	3	8	6
„	4	$1\frac{1}{4}$		„	2	9	3
„	6	$2\frac{1}{2}$		„	3	14	6
„	7	$3\frac{3}{4}$		„	4	7	9

6. Case VI. Learn by heart:

	s.	d.		s.	d.	
12 pence =	1	0	30 pence =	2	6	= half-a-crown.
18	1	6	36	3	0	
20	1	8	40	3	4	
24	2	0	48	4	0	

	s.	d.		s.	d.	
60 pence =	5	0	100 pence =	8	4	
70	5	10	120	10	0	
72	6	0	150	12	6	
80	6	8	180	15	0	
84	7	0	200	16	8	
90	7	6	240	20	0	= one pound.
96	8	0				

If a dozen things cost 1*s.*, each thing will cost 1*d.*; if 6*d.*, $\frac{1}{2}$*d.*; if 3*d.*, $\frac{1}{4}$*d.*; and if 9*d.*, $\frac{3}{4}$*d.*

Find the cost of one thing at 9*s.* a dozen. *Ans.* 9*d.*

Find the cost of one thing at £1 a dozen. *Ans.* 1*s.* 8*d.*, because £1 = 20*s.*, therefore each article will cost 20*d.*, and 20*d.* = 1*s.* 8*d.*

Find the cost of one article at 8*s.* 6*d.* a dozen. *Ans.* 8$\frac{1}{2}$*d.*, because 8*s.* 6*d.* is eight and a half shillings, therefore each article costs eight and a half pence.

Find the cost of one article at 15*s.* 9*d.* per dozen. *Ans.* 1*s.* 3$\frac{3}{4}$*d.*, because 15*s.* 9*d.* is fifteen and three-quarter shillings, therefore each article will cost fifteen and three-quarter pence.

Find the cost of one article at £1. 1*s.* 3*d.* a dozen. *Ans.* 1*s.* 9$\frac{1}{4}$*d.*, because £1. 1*s.* 3*d.* is twenty-one and a quarter shillings, therefore each article will cost twenty-one and a quarter pence, or 1*s.* 9$\frac{1}{4}$*d.*

Exercise M.

		£.	s.	d.			s.	d.
Find the cost of one thing at		0	5	9	per dozen.	*Ans.*	0	5$\frac{3}{4}$
"	"	0	10	3		"	0	10$\frac{1}{4}$
"	"	0	2	6		"	0	2$\frac{1}{2}$
"	"	0	15	6		"	1	3$\frac{1}{2}$
"	"	1	3	9		"	1	11$\frac{3}{4}$
"	"	0	16	3		"	1	4$\frac{1}{4}$
"	"	2	5	0		"	3	9
"	"	3	8	6		"	5	8$\frac{1}{2}$
"	"	4	7	9		"	7	3$\frac{3}{4}$
"	"	2	1	3		"	3	5$\frac{1}{4}$
"	"	4	13	6		"	7	9$\frac{1}{2}$
"	"	3	9	9		"	5	9$\frac{3}{4}$
"	"	1	6	3		"	2	2$\frac{1}{4}$
"	"	2	17	3		"	4	9$\frac{1}{4}$

	£.	s.	d.			s.	d.
Find the cost of one thing at	3	15	9	per dozen.	*Ans.*	6	$3\frac{3}{4}$
" "	4	8	6		"	7	$4\frac{1}{2}$
" "	4	18	9		"	8	$2\frac{3}{4}$
" "	3	11	6		"	5	$11\frac{1}{2}$
" "	2	14	3		"	4	$6\frac{1}{4}$

These six cases are intended to give the pupil facility in interconverting pounds, shillings, and pence. Other applications of the shillings and pence tables may be found equally serviceable, as, for instance, the calculation of interest at 5 per cent. per annum, which amounts to a payment of a shilling per pound each year, and a penny per pound each month.

Case I. Find the interest to be paid on a loan of £45 for one month. *Ans.* 3*s.* 9*d.*, because for one pound we pay one penny, therefore for £45 we pay 45 pence, or 3*s.* 9*d.*

Exercise N.

				s.	d.
Find the interest on	£70	for one month.	*Ans.*	5	10
"	50	"	"	4	2
"	66	"	"	5	6
"	94	"	"	7	10
"	100	"	"	8	4
"	37	"	"	3	1
"	120	"	"	10	0

Case II. Find the interest to be paid on a loan of £56 for one year. *Ans.* £2. 16*s.*, because for one pound we pay one shilling, therefore for £56 we pay 56 shillings, or £2. 16*s.*

Exercise O.

				£.	s.	d.
Find the interest on	£17	for one year.	*Ans.*	0	17	0
"	24	"	"	1	4	0
"	38	"	"	1	18	0
"	41	"	"	2	1	0
"	52	"	"	2	12	0
"	60	"	"	3	0	0
"	79	"	"	3	19	0
"	83	"	"	4	3	0

Exercise P.

			s.	d.
How many shillings are	14 pence?	*Ans.*	1	2
,,	21	,,	1	9
,,	32	,,	2	8
,,	87	,,	7	3
,,	79	,,	6	7
,,	53	,,	4	5
,,	46	,,	3	10
,,	95	,,	7	11
,,	41	,,	3	5
,,	12	,,	1	0
,,	23	,,	1	11
,,	78	,,	6	6
,,	97	,,	8	1
,,	35	,,	2	11
,,	60	,,	5	0
,,	59	,,	4	11
,,	48	,,	4	0

Exercise Q.

			£.	s.	d.
How many pounds are	91 shillings?	*Ans.*	4	11	0
,,	82	,,	4	2	0
,,	73	,,	3	13	0
,,	64	,,	3	4	0
,,	55	,,	2	15	0
,,	46	,,	2	6	0
,,	37	,,	1	17	0
,,	28	,,	1	8	0
,,	89	,,	4	9	0
,,	70	,,	3	10	0
,,	68	,,	3	8	0
,,	100	,,	5	0	0
,,	115	,,	5	15	0
,,	120	,,	6	0	0
,,	128	,,	6	8	0
,,	138	,,	6	18	0
,,	150	,,	7	10	0

CHAPTER III.

ADDITION.

1. CONCRETE. The word Addition is derived from the Latin *addo*, and means putting together.

2. Learn by heart: *This sign* (+) *is called* PLUS, *and means that the quantities between which it stands are to be added together.*

3. Learn by heart: *In Addition, the quantities to be added together are called the* ADDENDA, *and the answer is called the* SUM.

4. Add together £5, £4, £8, £7. *Ans.* £24.

Find the sum of 6*s.*, 5*s.*, 9*s.*, 4*s.*, 7*s.* *Ans.* 31*s.*, or £1. 11*s.*

Simplify 7*d.* + 4*d.* + 9*d.* + 6*d.* + 11*d.* *Ans.* 37 pence, or 3*s.* 1*d.*

Add together 8*s.* 3*d.* and 7*s.* 4*d.* *Ans.* 15*s.* 7*d.*

Add together 4*s.* 9*d.*, 5*s.* 4*d.*, 6*s.* 7*d.*, 9*s.* 3*d.*, and 2*s.* 8*d.* It is usual and convenient to write the addenda under one another, thus:

s.	*d.*
4	9
5	4
6	7
9	3
2	8

Let us begin with the shillings, as the more important. *Ans.* 26 shillings and 31 pence; but as 31 pence are 2*s.* 7*d.*, the handier answer is 28*s.* 7*d.*, or £1. 8*s.* 7*d.* From this it will be seen that labour would be saved by beginning with the pence, as the sum of the shillings can make no alteration in that of the pence, while the sum of the pence may yield some shillings.

Simplify 6*s.* 8*d.* + 4*s.* 3*d.* + 5*s.* 9*d.* + 3*s.* 11*d.* + 9*s.* 10*d.* + 1*s.* 2*d.*

Modus operandi:

s.	*d.*
6	8
4	3
5	9
3	11
9	10
1	2

Add up the pence column, 43 pence, which are 3*s.* 7*d.*; put down the 7*d.* under the pence, and add on or *carry* the 3*s.* to the shillings'

column, which now yields 31 shillings, or £1. 11*s.*; hence the total answer is £1. 11*s.* 7*d.*

Simplify £5. 6*s.* 0*d.* + £8. 3*s.* 0*d.* + £9. 7*s.* 0*d.* + £1. 4*s.* 0*d.* + £3. 9*s.* 0*d.* + £4. 8*s.* 0*d.*

Mod. op.:

£.	s.	d.
5	6	0
8	3	0
9	7	0
1	4	0
3	9	0
4	8	0

Add the shillings' column, which yields 37 shillings, or £1. 17*s.*; write the 17 shillings under the shillings, and *carry* the £1 to the pounds' column, which now yields £31; therefore the total answer is £31. 17*s.*

Find the sum of £6. 8*s.* 3*d.*, £2. 5*s.* 7*d.*, £3. 9*s.* 2*d.*, £5. 7*s.* 4*d.*, £9. 6*s.* 9*d.*

Mod. op.:

£.	s.	d.
6	8	3
2	5	7
3	9	2
5	7	4
9	6	9
26	17	1

Add the pence column, which yields 25 pence, or 2*s.* 1*d.*; write the 1*d.* under the pence column, and *carry* the 2*s.* to the shillings, which now yield 37 shillings, or £1. 17*s.*; write the 17 shillings under the shillings' column, and carry the pound to the pounds' column, which now yields £26; therefore the total answer is £26. 17*s.* 1*d.*, which is written under the line, as above.

Find the sum of $4\frac{3}{4}d.$, $5\frac{1}{2}d.$, $8\frac{1}{2}d.$, $9\frac{1}{4}d.$, and $10\frac{3}{4}d.$

Mod. op.:

	d.
	$4\frac{3}{4}$
	$5\frac{1}{2}$
	$8\frac{1}{2}$
	$9\frac{1}{4}$
	$10\frac{3}{4}$
3	$2\frac{3}{4}$

Add the farthings, counting the $\frac{1}{2}d.$ as two farthings; this yields 11 farthings, or $2\frac{3}{4}d.$; write the $\frac{3}{4}d.$ under the farthings, and carry

the $2d.$ to the pence, which now yield 38 pence, or $3s.$ $2d.$; therefore the answer is $3s.$ $2\frac{3}{4}d.$

Add £4. $3s.$ $7d.$, $4s.$ $5\frac{1}{2}d.$, £7. $8s.$ $8d.$, $9\frac{1}{2}d.$, £6. $0s.$ $3\frac{1}{4}d.$, £9. $5s.$ $0\frac{1}{2}d.$, £3. $0s.$ $6\frac{3}{4}d.$, $8s.$ $0\frac{3}{4}d.$, £8. $9s.$ $4\frac{3}{4}d.$

Mod. op.:

£.	s.	d.
4	3	7
	4	5½
7	8	8
		9½
6	0	3¼
9	5	0½
3	0	6¾
	8	0¾
8	9	4¾
39	0	10

Add the farthings: 16 farthings are $4d.$, which we carry to the pence, not writing anything under the farthings; add the pence: 46 pence are $3s.$ $10d.$, write $10d.$ under the pence, and carry $3s.$ to the shillings; add the shillings: $40s.$ are £2 exactly, therefore write 0 under the shillings, and carry the £2; add the pounds, which now yield £39, making the total £39. $0s.$ $10d.$

£.	s.	d.
3	7	4
5	2	8½
7	4	5¼
4	8	6¾
5	6	1½
4	0	9¾
7	2	11½
36	12	11¼

The following wording, *and no more*, is to be used: 2, 5, 7, 10, 11, 13 (farthings), 1, carry 3; 14, 23, 24, 30, 35, 43, 47 (pence), 11, carry 3; 5, 11, 19, 23, 25, 32 (shillings), 12, carry 1; 8, 12, 17, 21, 28, 33, 36 (pounds).

Exercise I.

(1)

£.	s.	d.
5	4	2
1	7	8

(2)

£.	s.	d.
5	3	4
8	2	6

(3)

£.	s.	d.
7	6	10
8	4	6

(4)

£.	s.	d.
7	8	11¾
5	6	7½

(5)

9	8	4
2	7	3
4	5	7

(6)

5	4	6¼
3	8	9¼
1	2	7½

(7)

7	2	3¼
8	4	2½
5	7	6¾

(8)

1	2	3
4	5	6
7	8	9
	4	5

(9)			(10)			(11)			(12)		
£.	s.	d.	£.	s.	d.	£.	s.	d.	£.	s.	d.
1	3	5½	7	7	7¼	3	7	4	7	6	5
5	8	9¾	8	9	10½	6	4	8¼	4	3	2½
7	5	3¼		4	5¼	4	9	2½	8	5	2
9	2	7	3	2	7	2	6	0½	9	6	9¼
									2	3	8

(13)			(14)			(15)			(16)		
8	7	9¾	1	2	4½	6	2	4½	2	6	9¾
8	7	9¾	4	8	3	6	5	9¼	3	7	11½
8	7	9¾	1	5	7¾	3	2	10½	4	8	10
8	7	9¾	5	7	7¼	7	0	0¾	5	9	9¼
8	7	9¾	9	1	7¼	8	8	8	6	5	8

(17)			(18)			(19)			(20)		
3	6	4	8	4	5	1	2	3¼	1	4	6¼
3	8	8	6	3	9	4	5	6½	9	6	7½
6	6	7	9	4	10¼	7	8	9	2	6	8¼
2	6	6	7	6	7½	9	6	3¾	6	8	4¼
9	4	1	5	5	5	6	8	5	3	6	9¼
7	9	9	3	7	8	7	4	1½	2	7	0

(21)			(22)			(23)			(24)		
4	7	7½	7	6	4¾	2	7	8	3	1	6¼
4	8	11½	8	6	3¾	5	9	3	7	4	5½
2	5	11½	5	9	4¼	4	2	11	2	7	9¾
7	8	9¼	4	7	11½	7	8	4	2	8	10¼
6	6	6¼	5	8	9¼	6	2	1	9	0	8¼
7	9	5¾	2	2	1½	1	0	6	3	4	11¾
						3	9	10	1	6	2¾

(25)			(26)			(27)			(28)		
1	3	9½	8	9	4	9	8	7½	6	7	9¼
2	7	2¾	2	5	7	6	5	4¼	1	8	10¼
3	6	4	8	6	3	3	2	1½	2	9	11½
	8	7¼	6	9	5	1	4	2¼	3	5	4¾
4	9	1¾	1	4	9	7	5	8¾	9	7	6¾
6	9	5½	3	8	10	3	8	4½	7	5	9¼
8	8	8	6	4	11	6	1	5¾	8	7	2½

(29)			(30)			(31)			(32)		
6	9	11¼	8	1	2	5	7	8	8	7	10½
4	9	4½	7	3	4	3	8	7½	5	4	4
2	6	6¾	6	5	9	7	7	9	6	2	2¼
3	9	9¾	6	5	4	2	8	9	7	9	9
8	7	2½	3	2	1	9	8	9	7	7	10¾
8	9	7¾	3	7	8	4	7	5¼	6	7	4
6	7	5½	2	1	11	5	9	4	8	6	5½
			1	4	10	8	8	3	4	3	2½

(33)

£.	s.	d.
1	9	7¼
2	0	8½
3	1	9¾
4	2	10¼
5	3	11½
6	4	1¾
7	5	2¼
8	6	3½

(34)

£.	s.	d.
7	8	7½
7	9	9
6	0	4
5	0	3
9	9	8
7	6	3
2	2	0¼
5	5	7½

(35)

£.	s.	d.
1	9	10¼
4	8	9½
2	7	11¼
3	8	2½
6	9	8¾
8	7	6¾
4	6	7¼
7	5	3

(36)

£.	s.	d.
6	8	4
6	9	5
7	3	10
4	3	11
7	7	7
6	8	4
3	2	7
1	1	1¼

(37)

5	5	3
7	6	8¼
4	9	7
9	9	4¼
2	3	5
6	0	7¼
7	7	7
8	7	7¼

(38)

4	7	3¼
6	5	9¼
8	8	10¾
2	2	4
1	6	11
3	5	5¾
5	9	1¼
7	8	2½

(39)

1	2	3½
4	5	6½
7	8	9½
1	0	11¼
1	2	1¾
4	5	6¼
5	6	7¼
3	6	9¾
7	6	5¾

(40)

1	0	2
2	2	5
3	4	8½
4	6	11
5	8	4
6	0	7
7	2	0¼
8	4	3
9	6	6

(41)

8	0	10½
7	0	3¼
9	0	5½
6	0	7¼
2	0	8½
2	0	9½
2	0	6¼
3	0	6¾
7	0	2¾

(42)

6	5	7½
	5	7½
1	7	5¼
7	6	11
	8	9
	4	8½
3	7	7
2	6	8
9	9	9

(43)

7	6	5½
8	8	10¼
3	6	11¾
6	8	10½
2	7	11¼
9	7	11¾
4	9	9½
1	4	7½
5	3	6¼

(44)

9	9	7½
7	6	4¼
3	7	6½
3	9	7½
6	2	5¼
3	2	3
7	6	5
4	0	0
9	9	9¼

(45)

3	2	0
1	8	9½
4	1	9¼
1	4	0½
5	2	9¾
9	8	7
2	7	6½
1	6	3
4	3	2¼

(46)

9	8	7¼
9	9	10¾
9	3	7¾
5	7	8¾
6	6	6½
7	5	2¼
3	0	4¼
4	2	2¼
2	9	11½
7	6	3¾

(47)

3	7	8¾
3	7	8¾
3	7	8¾
3	7	8¾
3	7	8¾
3	7	8¾
3	7	8¾
3	7	8¾
3	7	8¾
3	7	8¾

(48)

6	1	5
2	3	11
9	7	6
8	7	10
8	9	9
5	7	3
3	4	6
8	6	3
9	8	7
2	3	2

(49)			(50)			(51)			(52)		
£.	s.	d.	£.	s.	d.	£.	s.	d.	£.	s.	d.
1	6	$11\frac{1}{2}$	1	0	$9\frac{1}{4}$	7	6	$8\frac{1}{2}$	8	5	$3\frac{1}{4}$
2	7	$6\frac{3}{4}$	2	8	$7\frac{1}{2}$	1	5	$11\frac{3}{4}$	9	6	4
3	8	$5\frac{1}{4}$	3	6	$5\frac{3}{4}$	2	7	3	7	3	$11\frac{1}{2}$
4	1	$7\frac{1}{4}$	6	4	$11\frac{1}{4}$	3	8	$5\frac{1}{2}$	7	8	10
5	9	$4\frac{1}{2}$	4	9	$10\frac{3}{4}$	8	9	8	9	5	$6\frac{3}{4}$
6	2	$8\frac{3}{4}$	8	7	$8\frac{1}{2}$	9	1	$7\frac{1}{4}$	8	3	7
7	3	$3\frac{1}{4}$	9	4	$5\frac{1}{2}$	4	4	6	8	7	$7\frac{3}{4}$
8	5	$9\frac{3}{4}$	3	3	$2\frac{3}{4}$	6	2	$2\frac{3}{4}$	9	5	9
9	4	$2\frac{1}{2}$	7	9	$4\frac{1}{4}$	5	6	10	7	6	$4\frac{1}{2}$
9	0	$10\frac{1}{2}$	4	4	$6\frac{1}{2}$	5	3	8	7	8	6
						5	3	$1\frac{1}{4}$	9	3	$3\frac{1}{4}$
									6	2	$5\frac{1}{2}$

5. Add 13*s.*, 17*s.*, 19*s.*, 15*s.*, 12*s.*, 14*s.*, 16*s.*, 18*s.*

	s.
	13
	17
	19
	15
	12
	14
	16
	18
6	4

Wording: **8, 14, 18, 20, 25, 34, 41, 44 shillings, 4*s.*, carry 4 (half-sovs.); 5, 6, 7, 8, 9, 10, 11, 12 (half-sovs.), or £6. Total answer, £6. 4*s.***

Add £5. 17*s.* 0*d.*, £8. 13*s.* 0*d.*, £9. 11*s.* 0*d.*, £4. 15*s.* 0*d.*, £7. 13*s.* 0*d.*, £1. 10*s.* 0*d.*

£.	s.	d.
5	17	0
8	13	0
9	11	0
4	15	0
7	13	0
1	10	0
37	19	0

Wording: **0 pence; 3, 8, 9, 12, 19 (shillings), 9, carry 1 (half-sov.); 2, 3, 4, 5, 6, 7 (half-sovs.), 1, carry 3 (pounds); 4, 11, 15, 24, 32, 37. Total, £37. 19*s.* 0*d.***

Add £7. 18*s.* $6\frac{3}{4}$*d.*, £5. 19*s.* $10\frac{1}{2}$*d.*, £3. 11*s.* $4\frac{1}{4}$*d.*, £6. 13*s.* $8\frac{3}{4}$*d.*, £5. 8*s.* $7\frac{1}{2}$*d.*

£.	s.	d.
7	18	$6\frac{3}{4}$
5	19	$10\frac{1}{2}$
3	11	$4\frac{1}{4}$
6	13	$8\frac{3}{4}$
5	8	$7\frac{1}{2}$
29	12	$1\frac{3}{4}$

Wording: **2, 5, 6, 8, 11 (farthings), 3, carry 2; 9, 17, 21, 31, 37 (pence), 1, carry 3; 11, 14, 15, 24, 32 (shillings), 2, carry 3 (half-sovs.); 4, 5, 6, 7 (half-sovs.), 1, carry 3; 8, 14, 17, 22, 29. Total, £29. 12*s.* $1\frac{3}{4}$*d.***

Add £17. 14*s.* 3¼*d.*, £159. 11*s.* 5¾*d.*, £586. 16*s.* 11½*d.*, £83. 13*s.* 2*d.*, £36. 18*s.* 10½*d.*

£.	s.	d.
C. X. I.	£½. I.	
1 7	1 4	3¼
1 5 9	1 1	5¾
5 8 6	1 6	11½
8 3	1 3	2
3 6	1 8	10½
8 8 4	1 4	9

Wording: 2, 4, 7, 8, carry 2; 12, 14, 25, 30, 33, 9, carry 2; 10, 13, 19, 20, 24, 4, carry 2; 3, 4, 5, 6, 7, 1, carry 3; 9, 12, 18, 27, 34 (pounds), 4 (pounds) and carry 3 (tens of pounds); 6, 14, 22, 27, 28 (tens of pounds), 8 (tens of pounds), carry 2 (hundreds of pounds); 7, 8 (hundreds of pounds). Total, £884. 14*s.* 9*d.*

Add £5286. 16*s.* 8½*d.*, £397. 13*s.* 10¾*d.*, £6417. 18*s.* 5½*d.*, £96. 8*s.* 7¾*d.*, £12497. 16*s.* 8½*d.*, £638. 3*s.* 9*d.*, £8. 7*s.* 3¾*d.*

£.	s.	d.
XM. M. C. X. I.	£½. I.	
5 2 8 6	1 6	8½
3 9 7	1 3	10¾
6 4 1 7	1 8	5½
9 6	8	7¾
1 2 4 9 7	1 6	8½
6 3 8	3	9
8	7	3¾
2 5 3 4 3	5	5¾

Wording: 3, 5, 8, 10, 13, 15, 3, carry 3; 6, 15, 23, 30, 35, 45, 53, 5, carry 4; 11, 14, 20, 28, 36, 39, 45, 5, carry 4; 5, 6, 7, 8, carry 4; 12, 20, 27, 33, 40, 47, 53 (pounds), 3, carry 5 (tens of pounds); 8, 17, 26, 27, 36, 44 (tens of pounds), 4, carry 4 (hundreds of pounds); 10, 14, 18, 21, 23 (hundreds of pounds), 3, carry 2 (thousands of pounds); 4, 10, 15, 5, carry 1 (ten thousand pounds); 2. Total, £25343. 5*s.* 5¾*d.*

Add £7486. 15*s.* 10*d.*, £2593. 13*s.* 8½*d.*, £837. 11*s.* 10¾*d.*, £3596. 16*s.* 3½*d.*, £38. 5*s.* 11¼*d.*, £42,376. 7*s.* 7¾*d.*, £2000, £1120. 1*s.* 10*d.*, 7*s.* 11*d.*

£.	s.	d.
7486	15	10
2593	13	8½
837	11	10¾
3596	16	3½
38	5	11¼
42376	7	7¾
2000	0	0
1120	1	10
	7	11
60050	1	0¾

Wording: 3, 4, 6, 9, 11, 3, carry 2; 13, 23, 30, 41, 44, 54, 62, 72, 0, carry 6; 13, 14, 21, 26, 32, 33, 36, 41, 1, carry 4; 5, 6, 7, 8, carry 4; 10, 18, 24, 31, 34, 40, 0, carry 4; 6, 13, 16, 25, 28, 37, 45, 5, carry 4; 5, 8, 13, 21, 26, 30, 0, carry 3; 4, 6, 8, 11, 13, 20, 0, carry 2; 6. Total, £60,050. 1*s.* 0¾*d.*

Before dropping the cumbrous heading and wording adopted in the earlier cases, *several* sums should be worked *aloud*, till the pupil is perfectly familiar with the meaning of each column.

Exercise II.

(1)

£.	s.	d.
538	13	10½
8427	15	8¾
13	17	11
642	8	7½

(2)

£.	s.	d.
7040	19	1½
86	13	5¾
942	8	7¼
2568	10	10½

(3)

£.	s.	d.
42768	7	8½
13590	13	2¾
276	12	1¼
8402	5	8¾
35679	16	3¾

(4)

796	15	8¾
1248	19	11
2125	13	10¾
203	4	3¼
8751	0	1
2125	13	10¾

(5)

7684	7	8½
	15	3¼
19	13	10
256	15	11¾
1825	8	4¼
32769	12	9½
82103	1	7

(6)

786	13	10
5419	12	8
8	17	6
427	11	2
2040	13	3
966	16	6
7045	1	11

(7)

2197	12	10¼
1208	0	8½
319	14	11¾
9420	9	9¾
8531	16	7½
7642	1	2¼
6753	18	6¾
5864	3	5½

(8)

127556	11	4½
71042	2	3
48931	19	7¼
1632197	3	6½
49823	18	9¾
3286	5	8
254719	16	10½
6609	7	11½

(9)

416	13	11
5274	14	6
3708	5	8
2415	9	9
56780	17	3
26317	6	4
8412	18	10
90710	4	1
11768	12	2
1259	17	5
32708	8	7
673591	19	6

(10)

1678	18	11½
5246	17	10
356	13	5¾
8562	5	9
3427	6	0½
6	11	5¼
249	12	3½
8542	6	0¼
67	10	8¼
243	19	2½
14274	8	4¾
57343	9	10¾

By Dictation, if convenient:

(11) Add £539. 15*s.* 7*d.*, £18. 9*s.* 2¾*d.*

(12) Add £467. 10*s.* 9½*d.*, £8414. 3*s.* 7½*d.*, £6. 6*s.* 6½*d.*

(13) Add £558. 4*s.* 9¾*d.*, £25. 15*s.* 2¾*d.*, £532. 9*s.* 7*d.*, £7010. 11*s.* 4½*d.*

(14) Add £4279. 13*s.* 8½*d.*, £176. 15*s.* 9*d.*, £2040. 11*s.* 10¼*d.*, £1857. 16*s.* 9½*d.*, £855. 5*s.* 5¾*d.*

(15) Add £853. 12*s.* 9*d.*, £1866. 4*s.* 10*d.*, £851. 2*s.* 11*d.*, £2825. 8*s.* 4*d.*, £76,902. 11*s.* 3*d.*, £16,700. 19*s.* 11*d.*

(16) Add £9210. 3*s.* 7¾*d.*, £8127. 1*s.* 0*d.*, £8888. 0*s.* 11½*d.*, £53. 13*s.* 8½*d.*, £12,072. 3*s.* 1*d.*, £978. 16*s.* 3¼*d.*, £4063. 15*s.* 6½*d.*

(17) Add £4605. 1*s.* 10½*d.*, £2031. 15*s.* 3¾*d.*, £26,664. 12*s.* 10½*d.*, £161. 1*s.* 1½*d.*, £4024. 7*s.* 0¼*d.*, £2914. 4*s.* 4¼*d.*, £10,507. 17*s.* 7¾*d.*, £31,523. 12*s.* 11¼*d.*

(18) Add £1583. 11*s.* 3*d.*, £260. 2*s.* 4*d.*, £937. 17*s.* 10*d.*, £7064. 5*s.* 8*d.*, £525. 19*s.* 2*d.*, £348. 6*s.* 1*d.*, £69. 13*s.* 9*d.*, £8708. 4*s.* 11*d.*, £929. 18*s.* 7*d.*

(19) Add £91,021. 10*s.* 8¼*d.*, £28,943. 17*s.* 6¾*d.*, £3790. 2*s.* 4*d.*, £868. 15*s.* 9½*d.*, £5075. 9*s.* 10½*d.*, £6754. 16*s.* 5½*d.*, £538. 7*s.* 11*d.*, £4213. 12*s.* 2¾*d.*, £1967. 5*s.* 3*d.*, £2064. 13*s.* 7¼*d.*

(20) Add £63,185. 12*s.* 11½*d.*, £15,384. 10*s.* 8*d.*, £0. 8*s.* 10¼*d.*, £393,708. 6*s.* 5*d.*, £32,809. 4*s.* 3¾*d.*, £47. 2*s.* 4½*d.*, £27,512. 19*s.* 2½*d.*, £1760. 13*s.* 6*d.*, £3610. 7*s.* 7¾*d.*, £39,370. 11*s.* 1¾*d.*, £621,382. 5*s.* 0½*d.*, £3,531,658. 9*s.* 9¾*d.*

6. ABSTRACT.

Q. Add £5, £8, £7, £10.

A. £30.

Q. Add 5 sheep, 8 sheep, 7 sheep, 10 sheep.

A. 30 sheep.

Q. Add 5 things, 8 things, 7 things, 10 things.

A. 30 things.

Teacher. Therefore we say $5 + 8 + 7 + 10$ is 30, regardless of the name of the things counted. If the name is given with the number, the number is said to be *concrete;* if the number alone is given, it is said to be *abstract;* thus, 7 sheep, £9, 8 yards, 2 sounds, 3 ideas, 5 emotions, 4 times, are all *concrete* numbers; but the num-

bers 7, 9, 8, 2, 3, 5, 4, standing alone, are *abstract*. When in Ch. I. § 12, we said that 1000 = 10 times 100, &c., we were already using abstract numbers.

Q. Simplify 7 + 9.

A. 16.

Q. What do you mean by saying that 7 + 9 = 16?

A. 7 things of one kind added to 9 things *of the same kind* make 16 things of that kind.

Q. What are 7 horses and 9 paving-stones?

A. 7 horses and 9 paving-stones.

Teacher. We see, then, that we can only *add* quantities of the same kind.

Add 43, 178, 5297, 62,045, 19, 8, 7684, 5760, 112, and 7000.

Mod. op.:

```
   43
  178
 5297
62045
   19
    8
 7684
 5760
  112
 7000
-----
88146
```

Wording: 2, 6, 14, 23, 28, 35, 43, 46′, carry 4 (lay stress on this 6, and write it down in the act of pronouncing it); 5, 11, 19, 20, 24, 33, 40, 44′, carry 4; 5, 12, 18, 20, 21′, carry 2; 9, 14, 21, 23, 28′, carry 2; 8′.

7. Test of accuracy by casting out nines. (See Ch. II. § 4.)

Add 158, 6424, 543, 12764, 8248, 1251.

158	5
6424	7
543	3
12764	2
8248	4
1251	0
29388	3′

Cast out nines from each of the addenda, writing the remainders to the right of the vertical line, as above; cast out nines from these remainders, and also from the answer of the sum; if the results do

not agree, the answer cannot possibly be correct; but if they do agree, the chances are at least 8 to 1 in favour of its correctness.

For the reason of this process, see Ch. XI. § 9.

Wording: 1, 6, 14, **5′**; 6, 10, 1, 3, **7′**; 5, 9, 0, **3′**; 1, 3, 10, 1, 7, 11, **2′**; 8, 10, 1, 5, 13, **4′**; 1, 3, 8, 9, **0′**; 4, 6, 9, 0, 7, 12, *three;* 2, 5, 13, 4, 12, *three.*

Exercise III.

(1)	(2)	(3)	(4)
1768	2416	4907	8947
94	80519	356	397
550	9743	4520	8276
		38271	50703

(5)	(6)	(7)	(8)
4210	98376	650234	78903
349	4297	152467	4782
5827	376	24901	194763
631	4788	933378	4936
7856	76847	8426	531
		151	74267

(9)	(10)	(11)	(12)
265168	142857	9	48793561
521797	428571	87	20907528
293622	285714	654	7841
370448	857142	3210	25
452451	571428	98765	3069
538196	714285	432109	97856
554303	142857	8765432	867924
		10987654	1250708

(13)	(14)	(15)	(16)
987654321	9518582	16470431	8463243
987654321	829326	30	773904
987654321	70394	7180359	58654
987654321	6207	21642	327760
987654321	41574	85	9983
987654321	536342	3617	41732044
987654321	6019483	706506	75783
987654321	75620	482575	245477
987654321	8179	8429	3685473
		23869397	266599

(17)	(18)	(19)	(20)
8376214	6142	310531	8476231
5976414	250	973572	763249
635523	2035	861627	35604
46877	367	740537	9094
39256479	47498	810452	47776
99783	5809	168179	392448
246804	597	620438	7553210
8975578	196	975162	14578239
342429	6071	289705	278547
81660	38276	496561	6298
73431	3148	424280	17844
		983276	12345678

Exercise IV.

(1) John played at marbles and won on Monday 43 marbles, on Tuesday 101, on Wednesday 8, on Thursday 19, on Friday 119, and on Saturday 50. How many did he win in the week?

(2) In a certain school there are 67 boys, 59 girls, and 111 infants. How many pupils are there?

(3) The Books of Moses consist of 187 chapters, the Histories of 226, the Prophecies of 273, Job of 42, and the writings of David and Solomon of 201. How many chapters are there in the Old Testament?

The New Testament contains 260 chapters. How many chapters are there in the whole Bible?

(4) The battle of Thermopylæ took place 490 B.C., that of Bunker's Hill 1775 A.D. Find the time intervening between the two?

(5) How much money do I require to pay the following bills: butcher, £23. 7*s.* 6*d.*; baker, £9. 18*s.* 10*d.*; greengrocer, £1. 5*s.* 9*d.*; grocer, £7. 13*s.* 2*d.*; milkman, £2. 11*s.* 3*d.*; tailor, £15. 10*s.*; shoemaker, £7. 8*s.* 9*d.*; stationer, £4. 7*s.* 11*d.*; wine merchant, £8. 15*s.*?

(6) A cashier begins January with £48. 10*s.* 10*d.* in his till; there is paid him £75 in January, £120. 16*s.* 4*d.* in each of the next three months. How much will he then have to account for?

(7) A person left £3619. 6*s.* 8*d.* to each of his six children. Find the amount received by all of them.

(8) A farmer sold 2 oxen for £45. 12*s.* 6*d.*, a calf for £7. 15*s.*, 2 pigs for £12. 12*s.*, 4 sheep for £10, and 3 lambs at 17*s.* 9*d.* each. How many animals did he bring to market, and for how much did he sell them?

(9) A has 43 oxen, 145 sheep, 31 cows, and 19 horses; B has 57 lambs, 215 sheep, 8 horses, 7 pigs, and 10 calves; C has 60 lambs, 22 oxen, 89 sheep, and 12 calves; D has 67 cows, 28 horses, 11 pigs, and 3 lambs; E has 11 horses, 5 calves, 10 oxen, and 25 cows; F has 100 lambs, two herds of 37 oxen each, two flocks of sheep, one of 93 and the other of 39, and 50 pigs. How many animals are there of each kind, and how many altogether?

Exercise V.

(1)

£.	*s.*	*d.*
134	6	6
232	1	$11\frac{1}{4}$
1067	17	$9\frac{1}{2}$
4032	12	1
9416	9	$8\frac{1}{2}$
1067	13	8
1279	8	$7\frac{3}{4}$
4610	3	$2\frac{1}{4}$
752	15	$3\frac{1}{2}$
7187	10	$3\frac{1}{2}$
9312	8	$5\frac{1}{2}$

(2)

£.	*s.*	*d.*
13257	8	$11\frac{1}{2}$
3276	5	$9\frac{1}{4}$
46	3	$6\frac{3}{4}$
1287	14	$7\frac{1}{2}$
4917	10	8
147	0	$6\frac{3}{4}$
360	5	5
1379	17	$2\frac{3}{4}$
9	9	$10\frac{1}{2}$
1340	16	$9\frac{1}{4}$
906	10	$7\frac{1}{2}$
2222	5	1

(3)

73191211
31442376
16310732
6904109
432173041
92761528
13971140
105633198
78042
259766

(4)

5497530
5811756
761110
4028211
77436917
14376215
29351528
15286408
190589374
7319631
6911414
5016436
56616
1189401
7116917

CHAPTER IV.

SUBTRACTION.

1. The word Subtraction is derived from the Latin words *sub* and *traho*, and means drawing or taking away;—[cf. *summoveo*, to clear (a court).]

2. Learn by heart: *This sign* (—) *is called* MINUS, *and means that the quantity following it is to be subtracted or taken away from the quantity before it.*

3. Learn by heart: *In Subtraction, the quantity from which we subtract is called the* MINUEND, *the quantity to be subtracted is called the* SUBTRAHEND, *and the Remainder is called the* DIFFERENCE; e.g. if £7 be taken away from £11 the Remainder will be £4. Here £11 is the Minuend, £7 the Subtrahend, and £4 the Difference.

4. Take £8 from £13. *Ans.* £5.

Take 3*s.* 9*d.* from 7*s.* 11*d.*

Write them as in Addition.

$$\begin{array}{rr} 7 & 11 \\ 3 & 9 \\ \hline \end{array}$$

First take 3*s.* from 7*s.*, there remains 4*s.*, and the sum will so far stand thus:

$$\begin{array}{rr} 7 & 11 \\ 3 & 9 \\ \hline 4 & \end{array}$$

Now take 9*d.* from 11*d.*, there remains 2*d.*, so that the whole difference is 4*s.* 2*d.*

$$\begin{array}{rr} 7 & 11 \\ 3 & 9 \\ \hline 4 & 2 \end{array}$$

Take 6*s.* 8*d.* from 9*s.* 5*d.*:

$$\begin{array}{rr} 9 & 5 \\ 6 & 8 \\ \hline \end{array}$$

First take 6*s.* from 9*s.*, there remains 3*s.*, and the sum will so far stand thus:

$$\begin{array}{rr} 9 & 5 \\ 6 & 8 \\ \hline 3 & \end{array}$$

Q. Next take 8*d*. from 5*d*.
A. It can't be done.

Q. If I had 9*s*. 5*d*. in my purse, could I pay 6*s*. 8*d*.?
A. Yes; because 9*s*. 5*d*. is more than 6*s*. 8*d*.

Q. After paying the 6*s*., what will be left in my purse?
A. 3 shillings and 5 pence.
Q. How can I then pay the 8*d*.?
A. By changing one of the shillings into 12 pence.

Q. How many shillings will this now leave me?
A. Only 2 shillings.
Q. And how many pence?
A. 17 pence.
Q. Can I now pay the 8*d*.?
A. Yes.
Q. And how many pence will be left?
A. 9*d*.
Q. How much then is left altogether?
A. 2*s*. 9*d*.

Teacher. Thus the sum now stands:

$$\begin{array}{cc} 9 & 5 \\ 6 & 8 \\ \hline \cancel{3} & 9 \\ 2 & \end{array}$$

From £8. 5*s*. 1*d*. take £3. 7*s*. 9*d*.

$$\begin{array}{ccc} 8 & 5 & 1 \\ 3 & 7 & 9 \\ \hline 5 & & \end{array}$$

From £8 take £3, there remains £5.
Q. Next take 7*s*. from 5*s*.
A. It can't be done.
Q. How then shall I pay the 7*s*.?
A. Change one of the 5 pounds into 20 shillings.
Q. How many pounds will that leave me?
A. 4 pounds.

Teacher. Thus the sum now stands:

$$\begin{array}{rrr} \pounds 8 & 5 & 1 \\ 3 & 7 & 9 \\ \hline \cancel{8} & & \\ 4 & & \end{array}$$

Q. How many shillings shall I now have in my purse?

A. 25 shillings.

Q. If I pay the 7 shillings, how many shillings will be left?

A. 18 shillings.

Teacher. Thus the sum now stands:

$$\begin{array}{rrr} \pounds 8 & 5 & 1 \\ 3 & 7 & 9 \\ \hline \cancel{8} & 18 & \\ 4 & & \end{array}$$

Q. Now pay 9*d.*

A. Change one of the 18 shillings.

Q. How many shillings will remain?

A. 17 shillings.

Teacher. The sum now stands:

$$\begin{array}{rrr} \pounds 8 & 5 & 1 \\ 3 & 7 & 9 \\ \hline \cancel{8} & 1\cancel{8} & \\ 4 & 17 & \end{array}$$

Q. How many pence have I now?

A. 13 pence.

Q. Pay 9*d.*; remainder?

A. 4*d.*

Q. What then is the whole remainder?

A. £4. 17*s.* 4*d.*

Teacher. The sum now stands:

$$\begin{array}{rrr} \pounds 8 & 5 & 1 \\ 3 & 7 & 9 \\ \hline \cancel{8} & 1\cancel{8} & 4 \\ 4 & 17 & \end{array}$$

∴* the difference between £8. 5*s.* 1*d.* and £3. 7*s.* 9*d.* is £4. 17*s.* 4*d*

* ∴ = therefore.

Subtract £48. 17*s.* 9¾*d.* from £72. 3*s.* 11½*d.*

£.		*s.*		*d.*
X.	I.	£½.	I.	
7	2		$\cancel{3}$	11½
4	8	1	7	9¾
$\cancel{3}$	$\cancel{4}$	$\cancel{1}$	6	$\cancel{2}$¾
2	3			1

First pay 4 ten-pound notes out of 7 ten-pound notes; there remain 3 ten-pound notes; next we have to pay £8 out of £2, for which we must change one of the 3 ten-pound notes, leaving only 2 ten-pound notes. The ten-pound note changed, together with the £2 we had at first, yield £12, and paying £8, we have left £4. Next we have to pay one half-sovereign out of none, for which we must change one of the 4 pounds, leaving only £3; that £1 yields 2 half-sovereigns, and paying one of these leaves us 1 half-sovereign. Next we have to pay 7 shillings out of 3 shillings, for which we must change the half-sovereign; and this, together with the 3*s.* we had at first, yields 13*s.*; paying 7*s.* out of it, we have left 6*s.* Next we have to pay 9*d.* out of 11*d.*, and we have 2*d.* left. Lastly, we have to pay 3 farthings out of a halfpenny, for which we must change one of the 2 pence, leaving only 1*d.*, and yielding, with the halfpenny we had at first, 6 farthings, out of which we can pay 3 farthings, and have 3 farthings left. The total difference accordingly is £23. 6*s.* 1¾*d.**

This method of subtraction is always applicable; but to avoid erasures, it is better to begin with the lowest instead of the highest denomination. Thus:

£.		*s.*		*d.*
X.	I.	£½.	I.	
7·	2·	·	3	11·½
4	8	1	7	9 ¾
2	3		6	1 ¾

First pay ¾*d.* out of ½*d.*; it cannot be done; therefore change 1*d.*

* The object of this method of teaching is to lead the pupil along some such road as must have been traversed by the original inventors of the different Rules in arriving at the most concise processes, thus attaching the new notions to be acquired to those previously existing in the mind of the learner, or, in other words, *educing* the new ideas out of old ones, which is true *education.*

of the 11*d.*, leaving 10*d.*, and, to remind ourselves that we have done so, make a dot by the side of the 11*d.*; this penny changed, together with the halfpenny we had at first, yields 6 farthings, out of which we pay ¾*d.* and have ¾*d.* left. Next pay 9*d.* out of 10*d.*, which leaves 1*d.* Next pay 7 shillings out of 3 shillings; to do so, change one of the 2 pounds (putting the dot by the two) into 2 half-sovereigns; leave one of these in the half-sovereign column, and indicate by the dot that the other has been taken away and changed into 10*s.*; this, together with the 3*s.*, yields 13*s.*; pay the 7*s.*, there remain 6*s.* Next pay one half-sovereign, and no half-sovereigns are left. Next pay £8 out of £1, for which change one of the 7 ten-pound notes, leaving 6, and yielding, together with the £1, £11. Now pay the £8, leaving £3. Lastly, pay the 4 ten-pound notes out of the 6, and 2 ten-pound notes are left. Total remainder or difference, as before, £23. 6*s.* 1¾*d.*

From £1000, take £37. 8*s.* 6*d.*

£.				*s.*		*d.*
M.	C.	X.	I.	£½.	I.	
1·	0·	0·	0·	·	0·	0
		3	7		8	6
	9	6	2	1	1	6

6*d.* cannot be taken from 0*d.*, therefore we must change some higher coin or note. The only money we have is 1 thousand-pound note, which we change into 10 hundreds, putting a dot as before after the 1; we take now one of these 10 hundreds to the tens, leaving 9 hundreds in the hundreds' column, which is again indicated by a dot; this hundred gives ten tens, of which we take one, leaving 9 in the tens' column; this ten gives £10, of which we take £1, leaving 9 in the pounds' column; this £1 gives 2 half-sovereigns, of which we take one, leaving 1 under the half-sovereign column; the 1 we take gives 10 shillings, of which we take 1 shilling and leave 9; this shilling gives 12 pence. 6*d.* from 12*d.* leaves 6*d.*; 8*s.* from 9*s.* leaves 1*s.*; nothing from 1 half-sovereign leaves 1 half-sovereign; £7 from £9 leaves £2; 3 ten-pound notes from 9 ten-pound notes leaves 6 ten-pound notes; and nothing from 9 hundred-pound notes leaves 9 hundred-pound notes. Total remainder, £962. 11*s.* 6*d.*

EXAMPLES.

£.	s.	d.		Wording:
8	7	9		6 from 9, 3′
1	5	6		5 ,, 7, 2′
7	2	3		1 ,, 8, 7′

£.	s.	d.	Wording
£8·	·5·	3	7 from 15 (.) 8′
3	9	7	9 ,, 14 (..) 5′
4	15	8	0 ,, 1 1′
			3 ,, 7 4′

£.	s.	d.	Wording
£4·2 9·	1·3·	2·¼	3 from 5 (.) ½′d.
176	18	9 ¾	9 ,, 13 (.) 4′
252	14	4 ½	8 ,, 12 (.) 4′
			1 ,, 2 (.) 1′
			6 ,, 8 2′
			7 ,, 12 (.) 5′
			1 ,, 3 2′

£.	s.	d.	Wording
£6·0·5·0·	4·	0·	1 from 4 (..) 3′
1768	12	2 ¼	2 ,, 11 9′
4281	11	9 ¾	2 ,, 3 1′
			1 ,, 2 (..) 1′
			8 ,, 9 1′
			6 ,, 14 (..) 8′
			7 ,, 9 2′
			1 ,, 5 4′

£.	s.	d.	Wording
£2 7·0·0·	0·	0	7 from 12(....)5′
183	3	7	3 ,, 19* 16′
2516	16	5	3 ,, 9 6′
			8 ,, 9 1′
			1 ,, 6 5′
			0 ,, 2 2′

From £87,694. 0*s.* 0*d.*, subtract £24,132. 0*s.* 0*d.*

£.	s.	d.	Wording:
87694	0	0	2 from 4, 2′
24132	0	0	3 ,, 9, 6′
63562	0	0	1 ,, 6, 5′
			4 ,, 7, 3′
			2 ,, 8, 6′

* Where there are no half-sovereigns in the minuend, it is better to call £1 20 shillings, instead of 2 half-sovereigns.

From £42,618, take £16,942.

4·2·6·1 8	*Wording:*	2	from	8	6
1 6 9 4 2		4	,,	11 (.)	7
2 5 6 7 6		9	,,	15 (.)	6
		6	,,	11 (.)	5
		1	,,	3	2

From 90,017 units, take 18,659 units.

9·0·0·1·7	*Wording:*	9	from	17 (.)	8
1 8 6 5 9		5	,,	10 (...)	5
7 1 3 5 8		6	,,	9	3
		8	,,	9	1
		1	,,	8	7

From 1,000,000, take 63,497.

1·0·0·0·0·0·0	*Wording:*	7	from	10 (......)	3
6 3 4 9 7		9	,,	9	0
9 3 6 5 0 3		4	,,	9	5
		3	,,	9	6
		6	,,	9	3
		—		—	9

5. Test of accuracy by casting out nines. (See Ch. II. § 4, and Ch. III. § 7.)

143549	8
30718	1
112831	7′

Wording: 1, 5, 8, 13, 4, 8′; 3, 10, 1, 2, 10, 1′; 1 from 8, *seven;* 1, 2, 4, 12, 3, 6, *seven.*

1030412	2
764564	5
265848	6′

Wording: 1, 4, 8, 9, 0, 2′; 7, 13, 4, 8, 13, 4, 10, 1, 5′ (now as 5 cannot be taken from 2, add 9 to the 2, making 11); 5 from 11, *six;* 2, 8, 13, 4, 12, 3, 7, 15, *six.*

314254	1
123456	3
190798	7′

Wording: 3, 4, 8, 10, 1, 6, 10, 1′; 1, 3, 6, 10, 1, 6, 12, 3′; 3 from 10, *seven;* 1, 8, 16, *seven.*

4680135	0
3216540	3
1463595	6′

Wording: 4, 10, 1, 9, 0, 1, 4, 9, 0′; 3, 5, 6, 12, 3, 8, 12, 3′; 3 from 9, *six;* 1, 5, 11, 2, 5, 10, 1, *six.*

EXERCISE VI.

(1)

£.	s.	d.
778	4	5
536	2	4

(2)

£.	s.	d.
9999	19	$11\frac{3}{4}$
9876	5	$4\frac{3}{4}$

(3)

46596	17	10
13282	12	4

(4)

8712	15	3
930	8	7

(5)

4897	7	7
1624	8	11

(6)

8713	18	$9\frac{1}{4}$
1784	19	$11\frac{3}{4}$

(7)

7638	14	$3\frac{1}{2}$
147	17	$4\frac{1}{2}$

(8)

72312	5	$9\frac{3}{4}$
16743	8	$10\frac{1}{2}$

(9)

9472	4	$5\frac{3}{4}$
8763	15	$4\frac{1}{4}$

(10)

81403	15	$7\frac{1}{2}$
68	0	11

(11)

346004	1	$2\frac{1}{2}$
76802	14	$0\frac{1}{4}$

(12)

49060	12	$1\frac{3}{4}$
893	15	$3\frac{3}{4}$

(13)

130764	11	3
68943	13	$6\frac{1}{2}$

(14)

40076	0	$5\frac{3}{4}$
16307	9	$8\frac{1}{4}$

(15)

70	10	$0\frac{1}{2}$
6	16	$8\frac{3}{4}$

(16)

382	0	$0\frac{1}{4}$
187	0	$11\frac{1}{2}$

(17)

1000	0	0
999	19	$11\frac{3}{4}$

(18)

53761	0	0
7777	7	7

(19)

500	0	0
367	18	$10\frac{1}{4}$

(20)

8000	0	0
796	17	$2\frac{1}{2}$

EXERCISE VII.

(1)	(2)	(3)
734968	625137	435768
123454	214021	123989

(4)	(5)	(6)
243617	310703	57345
76329	25844	16462

(7)	(8)	(9)
3467825	3087902	1000000
178062	396999	999999
(10)	(11)	(12)
376462	857142	401050
84752	428571	123456
(13)	(14)	(15)
876543	543645	283461
777777	137968	99999
(16)	(17)	(18)
194580	422060	1147368
63245	234681	654891

(19)	(20)
5000807	100100
283876	11111

EXERCISE VIII.

By Dictation, if convenient.

(1) Take 196,734 from 321,708; also £4105. 13*s.* 8$\frac{3}{4}$*d.* from £12,019. 16*s.* 2$\frac{1}{2}$*d.*

(2) Find the difference between 1,178,458 and 1,780,412; also between £50,100. 2*s.* 2*d.* and £16,872. 13*s.* 10$\frac{1}{2}$*d.*

(3) Find the excess of £105. 3*s.* 0$\frac{1}{4}$*d.* over £83. 16*s.* 9$\frac{1}{2}$*d.*; also of 7015 over 2406.

(4) How much must I add to £13,502. 9*s.* 3*d.* to get £40,000; also to 62,318 to get 100,000?

(5) What must I take away from £11,111. 11*s.* 1$\frac{1}{4}$*d.* to leave £3805. 16*s.* 2$\frac{1}{2}$*d.*; also from 1,010,101 to leave 909,909.

(6) The latitude of London is 51$\frac{1}{2}^0$ N.; that of the Tropic of Cancer is 23$\frac{1}{2}^0$ N. How many degrees is London north of this Tropic?

(7) If I gain £37,105. 8*s.* 3$\frac{1}{2}$*d.* and lose £50,602. 9*s.* 10*d.*, how much shall I have gained or lost altogether?

(8) In a certain school there are 102 pupils, of whom 67 are boys and the remainder girls. How many girls are there?

(9) A father is 42 years of age, and his eldest son was born when he was 26 years old. How old is the son now? Also, by how

much is the father older than the son, and how much older than the son will the father be 12 years hence?

(10) It is now eleven o'clock a.m. What o'clock was it 6, 12, 15, and 24 hours ago?

6. To how many persons can I give 8*s.* 3*d.* if I have £2. 1*s.* 3*d.*?

£2	1	3	
	8	3	1
1	13	0	
	8	3	1
1	4	9	
	8	3	1
	16	6	
	8	3	1
	8	3	
	8	3	1
	—	—	5 persons.

To how many persons can I give £53. 12*s.* 9½*d.* if I have £492. 8*s.*?

£492	8	0	
53	12	9½	1
438	15	2½	
53	12	9½	1
385	2	5	
53	12	9½	1
331	9	7½	
53	12	9½	1
277	16	10	
53	12	9½	1
224	4	0½	
53	12	9½	1
170	11	3	
53	12	9½	1
116	18	5½	
53	12	9½	1
63	5	8	
53	12	9½	1
9	12	10½	9 persons, and £9. 12*s.* 10½*d.* over.

EXERCISE IX.

(1) To how many persons can I give £19. 16*s*. $3\frac{3}{4}$*d*. each, if I have £138. 14*s*. $2\frac{1}{4}$*d*?

(2) How many shares can I buy for £1120. 10*s*., if each costs £93. 7*s*. 6*d*.?

(3) How many times can I subtract £8. 9*s*. 4*d*. from £67. 14*s*. 8*d*.?

(4) How many heaps of 56 marbles each can I make out of 500 marbles?

(5) If I travel 144 miles a day, how many days shall I take to travel 1008 miles?

(6) How many strips of carpet, each 17 yards long, can I get out of 160 yards of carpet?

EXERCISE X.

Miscellaneous Examples on the preceding chapters.

(1) *a*. Read off: LVIII, LXIX, CXLVII, CCXCIV, MXC, MDCCCIV, MDXIX, MC, DCCCCIX, MDCCCLXIX.

b. Write in Roman Notation: Six; four; eleven; twenty-nine; one hundred and forty-four; six hundred and sixty-six; one thousand, two hundred and two; eleven hundred and fourty-four; twelve hundred and ninety-nine; one thousand, two hundred and thirty-four.

(2) *a*. Read, or write in words: 1008; 13015; 7012009; 4226843; 60606060; 987654321; 42105; 24150; 116001; 110061.

b. Write with Arabic Notation in the Decimal Scale: Five thousand and forty; twelve thousand and twelve; twelve hundred and twelve; twelve thousand, twelve hundred and twelve; thirteen millions, fourteen thousand and fifteen; fifty-eight millions; twenty millions, three hundred and sixty-five thousand and nineteen; fifty thousand and fifty; five hundred thousand and fifty; one million, one thousand and ten.

(3) Analyse (aloud or in writing) 3052604.

(4) Alfred the Great died at the age of 52, A.D. 901. In what year was he born?

(5) William the Conqueror began to reign A.D. 1066, and reigned 21 years. In what year did he die?

(6) A travels northwards 203 miles, B travels 167 miles in the same direction. How far will they be apart?

(7) If B had travelled southwards, how far apart would they be?

(8) I am now (A.D. 1869) 37 years old, and the battle of Trafalgar happened 27 years before I was born. In what year was it fought?

(9) Add the sum of 1008 and 639 to their difference.

(10) Take the difference of 1001 and 999 from their sum.

(11) A cashier receives on Monday, £596. 13*s.* 8*d.*; on Tuesday, £932. 11*s.* 4*d.*; on Wednesday, £403. 6*s.* 4*d.*; on Thursday, £67. 8*s.* 8*d.*; on Friday, £145. 17*s.* 6*d.*; and on Saturday, £854. 2*s.* 6*d.* His expenditure is on Monday, £139. 19*s.* 11*d.*; on Tuesday, £369. 8*s.* 10*d.*; on Wednesday, £860. 0*s.* 1*d.*; on Thursday, £630. 11*s.* 2*d.*; on Friday, nothing; on Saturday, £319. 9*s.* 9*d.* Find the balance in hand at the end of each day of the week.

(12) My income is £500 a-year. I spend £14. 3*s.* 6*d.* each quarter for rent; £3. 10*s.* 10½*d.* each quarter for taxes; £9. 3*s.* 4*d.* each half-year for insurance; £156 per annum for food, &c.; £7. 4*s.* for coals during the winter, and £1. 16*s.* during the summer; and £33. 6*s.* 8*d.* a-year schooling for *each* of three children. How much a-year is left me for other purposes?

(13) If I spend £2. 17*s.* 4*d.* a-week, having £16 on leaving home, how much shall I bring back after 5 weeks' holiday?

(14) If I spend every week 15*s.* 9*d.* for lodging; £1. 1*s.* for board; 4*s.* 6*d.* for travelling; 8*s.* 4*d.* for sundries; how long will £15 last me, and how much shall I be in debt at the end of 7 weeks?

(15) How many months of 30 days each are there in the 365 days of the year?

(16) An army consisted of 50,000 soldiers before the battle. The list of casualties was as follows: killed, 3,768 men and 419 officers; wounded, 9,483 men and 2,716 officers; missing, 802 men and 1 officer. What was the strength of the army after the battle?

(17) On the 1st of January I bought goods for £50. 18*s.* 6*d.*, and paid £10. 10*s.* on account. The balance is to be paid off in monthly instalments of £5. 15*s.* 6*d.* each. On what day shall I pay the last instalment?

(18) A clock and a watch are set at noon on Monday; the clock gains 4 minutes in each of its days (of 24 hours), the watch loses 2 minutes in the same time; what time will the watch indicate on Saturday when it is noon by the clock?

(19) If the clock gains 7 minutes and the watch gains 11 minutes each day by the clock, what time will the watch indicate on Saturday when it is noon by the clock?

(20) If I have £80 in the bank, and put by £52. 10*s.* a year, how long shall I be in accumulating £500?

(21) How many times must 6798 be added to 9212 to make 50,000?

(22) I have the following bills to pay: butcher, £14. 7*s.* 11*d.*; baker, £5. 9*s.* 8*d.*; grocer, £9. 10*s.* 10*d.*; greengrocer, 17*s.* 7*d.*; milkman, £1. 2*s.* 3*d.* How much shall I have left out of £35?

(23) A house with fixtures and furniture is bought for £1050; the price of the furniture is £335, that of the fixtures is £27. 10*s.* What was the sum paid for the house?

(24) John and Tom play at marbles; John begins with 158 marbles, and Tom with 271; Tom loses 56 marbles. Which has more, and by how much?

(25) At an election there were three polling-places. At the first, the Whig candidate obtained 766 votes, and the Tory 695; at the second, the Tory had 523 and the Whig 419; at the third, the Whig had 812 and the Tory 811. Which candidate was elected, and what was his majority?

7. By the three processes, Numeration, Addition and Subtraction, which have now been taught, all or nearly all arithmetical questions can in theory be solved. But in many cases the operation would be so long as to render the solution an impossibility in practice. The Chapters which follow deal with *Contractions of these processes.*

CHAPTER V.

MULTIPLICATION.

1. Find the amount of money contained in six bags if each holds £7. 8*s.* 9*d.*

£.	s.	d.
7	8	9
7	8	9
7	8	9
7	8	9
7	8	9
7	8	9
44	12	6

Instead of writing down £7. 8*s.* 9*d.* six times, we indicate this REPETITION thus : £7. 8*s.* 9*d.* $\times$ 6, read £7. 8*s.* 9*d.* *multiplied by* 6. If we know, without actually adding, that six TIMES nine are fifty-four, that six times eight are forty-eight, and that six times seven are forty-two, we can shorten the work thus: £7. 8*s.* 9*d.* $\times$ 6 $=$ £44. 12*s.* 6*d.*

Wording: 6 times 9 pence are 54 pence, put down 6 pence and carry 4 shillings; 6 times 8 shillings are 48 shillings, which, with the 4 shillings carried, make 52 shillings, put down 12 shillings and carry £2; six times £7 are £42, and £2 make £44. Total, £44. 12*s.* 6*d.*

2. Learn by heart: *This sign* ($\times$) *is called* MULTIPLIED BY, *and means that the number of things before it is to be* REPEATED AS MANY TIMES *as is indicated by the number following it.*

The word Multiplication is derived from the Latin *multi-plex,* many-fold.

3. Learn by heart: *The number or quantity which is to be multiplied is called the* MULTIPLICAND, *the number by which we multiply is called the* MULTIPLIER, *and the result is called the* PRODUCT. *The Multiplier and Multiplicand are often also called* FACTORS *of the Product.*

4. $3 \times 4 = 12$; $4 \times 3 = 12$; $\therefore 3 \times 4 = 4 \times 3$. The following demonstration shews that this holds true of any two numbers whatever:

. . . .
. . . .
. . . .

These dots read horizontally yield three fours, and read vertically they yield four threes. Similarly, seven columns of eight dots each would form eight lines of seven dots each, and so on with any other two numbers. Hence we can extend the meaning of the sign × when placed between two abstract numbers.

Learn by heart: *The sign* (×) *indicates that the number on either side of it is to be* REPEATED AS MANY TIMES *as is indicated by the number on the other side of it.*

Even if one of the numbers be concrete, say £7. 8*s*. 9*d*., it is optional to indicate the multiplication by 6, either by £7. 8*s*. 9*d*. × 6 (£7. 8*s*. 9*d*. *multiplied by* 6), or by 6 × £7. 8*s*. 9*d*. (6 *times* £7. 8*s*. 9*d*.).

5. From what has been said in § 1 of this Chapter, it follows that a certain series of multiplications already performed must be learnt by heart. This series is called the MULTIPLICATION TABLE.*

6. Learn by heart:

1 × 2 = 2 (read, once two is two).
2 × 2 = 4 (twice two is four).
3 × 2 = 6 (three times two is six).
4 × 2 = 8
5 × 2 = 10
6 × 2 = 12
7 × 2 = 14
8 × 2 = 16
9 × 2 = 18
10 × 2 = 20

Also:

2 × 1 = 2 (read, twice one is two).
2 × 2 = 4 (twice two is four).
2 × 3 = 6 (twice three is six).
2 × 4 = 8
2 × 5 = 10
2 × 6 = 12
2 × 7 = 14
2 × 8 = 16
2 × 9 = 18
2 × 10 = 20

Multiply £487. 9*s*. 10*d*. by 2.

£.	s.	d.
487	9	10
		2
974	19	8

Wording: 20 (pence), 8′, carry 1; 18, 19′ (shillings); 14′, carry 1; 16, 17′, carry 1; 8, 9′.

* For the full analysis of the Multiplication Table, see the A B C of Arithmetic, Part II., Teacher's Copy, pp. 39 to 73.

Double £7468. 18*s.* 11¾*d.*

£.	s.	d.
7468	18	11¾
		2
14937	17	11½

Wording: 6, ½′*d.*, carry 1; 22, 23, 11′ (pence), carry 1; 16, 17′ (shillings), carry 1; 2, 3 (half-sovs.), 1′, carry 1; 16, 17′, carry 1; 12, 13′, carry 1; 8, 9′; 14′.

Find twice 4,805,639.

```
4805639
      2
-------
9611278
```

Wording: 18′ carry 1; 6, 7′; 12′, carry 1; 10, 11′; 16′, carry 1; 8, 9′.

Multiply £3,276,003. 17*s.* 10¾*d.* by 2.

£.	s.	d.
3276003	17	10¾
		2
6552007	15	9½

Wording: 6, ½′*d.*, carry 1; 20, 21, 9′, carry 1; 14, 15′, carry 1; 2, 3, 1′, carry 1; 6, 7′; 0′; 0′; 12′, carry 1; 14, 15′, carry 1; 4, 5′; 6′.

Exercise XI. (*a*).

Apply this table to the following:

(1) How much money will there be in 2 bags, if each contains £245. 13*s.* 7¼*d.*?

(2) If I travel from London to Liverpool and back, the distance between them being 192 miles, how many miles shall I have travelled?

(3) How many soldiers are there in 2 regiments of 876 soldiers each?

(4) Find the cost of a pair of ponies costing £15. 17*s.* 6*d.* each?

(5) A travels 487 miles north, and B the same distance south. How far will they be apart?

(6) Find the double of £2493. 15*s.* 9¾*d.*

(7) 416296 × 2
(8) £40529. 6*s.* 0½*d.* × 2
(9) 278409 × 2
(10) £51732. 17*s.* 2¾*d.* × 2
(11) 301620 × 2
(12) £23154. 8*s.* 8¾*d.* × 2
(13) 510847 × 2
(14) £34088. 13*s.* 9½*d.* × 2
(15) 639756 × 2
(16) £62977. 5*s.* 3¼*d.* × 2
(17) 847538 × 2
(18) £79863. 19*s.* 10*d.* × 2
(19) 925317 × 2
(20) £18641. 7*s.* 11¼*d.* × 2

7. Learn by heart:	Also:
1 × 3 = 3 (read, once three is three).	3 × 1 = 3 (read, three times one is three).
2 × 3 = 6 (twice three is six).	3 × 2 = 6 (three times two is six).
3 × 3 = 9 (three times three is nine).	3 × 3 = 9 (three times three is nine).
4 × 3 = 12	3 × 4 = 12
5 × 3 = 15	3 × 5 = 15
6 × 3 = 18	3 × 6 = 18
7 × 3 = 21	3 × 7 = 21
8 × 3 = 24	3 × 8 = 24
9 × 3 = 27	3 × 9 = 27
10 × 3 = 30	3 × 10 = 30

Notice that the sum of the digits in the successive answers follows the order 3, 6, 9.

Exercise XI. (*b*).

(1) How much money will there be in 3 bags, if each contains £567. 18*s.* 11¼d.?

(2) Find the length of the 3 sides of a triangle, if each side is 493 feet long?

(3) How much do I require to give £2045. 7*s.* 6*d.* to each of 3 persons?

(4) How many soldiers in 3 armies of 389,012 men each?

(5) Find the value of 3 East-Indiamen, if each is worth £645,900.

(6) What was a man's fortune if he left £66,666. 13*s.* 4*d.* to each of his 3 children?

(7) 416296 × 3
(8) £40529. 6*s.* 0½*d.* × 3
(9) 278409 × 3
(10) £51732. 17*s.* 2¾*d.* × 3
(11) 301620 × 3
(12) £23154. 8*s.* 8¾*d.* × 3
(13) 510847 × 3
(14) £34088. 13*s.* 9½*d.* × 3
(15) 639756 × 3
(16) £62977. 5*s.* 3¼*d.* × 3
(17) 847538 × 3
(18) £79863. 19*s.* 10*d.* × 3
(19) 925317 × 3
(20) £18641. 7*s.* 11¼*d.* × 3

8. Learn by heart:	Also:
1 × 4 = 4 (read, once four is four).	4 × 1 = 4 (read, four times one is four).
2 × 4 = 8 (twice four is eight).	4 × 2 = 8 (four times two is eight).
3 × 4 = 12 (three times four is twelve).	4 × 3 = 12 (four times three is twelve).
4 × 4 = 16	4 × 4 = 16
5 × 4 = 20	4 × 5 = 20
6 × 4 = 24	4 × 6 = 24
7 × 4 = 28	4 × 7 = 28
8 × 4 = 32	4 × 8 = 32
9 × 4 = 36	4 × 9 = 36
10 × 4 = 40	4 × 10 = 40

EXERCISE XI. (*c*).

(1) How much money in 4 bags, if each contains £765. 0*s*. 11*d*.?

(2) Find the length of the sides of a square, if each is 894 feet long.

(3) How much do I owe altogether, if to each of 4 men I owe £321. 19*s*. 10*d*.?

(4) How many students in 4 colleges, if there are 465 in each?

(5) 1 lb. Troy has 5760 grains. How many grains in 4 lbs.?

(6) 1 cwt. (hundredweight) has 4 qrs. (quarters), and each qr. has 28 lbs. How many lbs. in a cwt.?

(7) 416296 × 4
(8) £40529. 6*s*. 0½*d*. × 4
(9) 278409 × 4
(10) £51732. 17*s*. 2¾*d*. × 4
(11) 301620 × 4
(12) £23154. 8*s*. 8¾*d*. × 4
(13) 510847 × 4
(14) £34088. 13*s*. 9½*d*. × 4
(15) 639756 × 4
(16) £62977. 5*s*. 3¼*d*. × 4
(17) 847538 × 4
(18) £79863. 19*s*. 10*d*. × 4
(19) 925317 × 4
(20) £18641. 7*s*. 11¼*d*. × 4

9. Learn by heart:

1 × 5 = 5 (read, once five is five).
2 × 5 = 10 (twice five is ten).
3 × 5 = 15 (three times five is fifteen).
4 × 5 = 20
5 × 5 = 25
6 × 5 = 30
7 × 5 = 35
8 × 5 = 40
9 × 5 = 45
10 × 5 = 50

Also:

5 × 1 = 5 (read, five times one is five).
5 × 2 = 10 (five times two is ten).
5 × 3 = 15 (five times three is fifteen).
5 × 4 = 20
5 × 5 = 25
5 × 6 = 30
5 × 7 = 35
5 × 8 = 40
5 × 9 = 45
5 × 10 = 50

Notice that the figure in the units' place is always either 5 or 0.

EXERCISE XI. (*d*).

(1) How much money in 5 bags, if each contains £4039. 18*s*. 7¾*d*.?

(2) How many petals are there in 376 forget-me-nots?

(3) What would be my income in 5 years at £2765. 10*s*. 10*d*. a-year?

(4) Find the length of the sides of a pentagon, if each is 137 inches long.

(5) A spends 5 times as much as B, whose yearly outlay is £941. 7*s.* 4*d.* Find A's expenditure.

(6) How far will a wheel 5 yards in circumference travel in making 3068 turns?

(7) 416296 × 5
(8) £40529. 6*s.* 0½*d.* × 5
(9) 278409 × 5
(10) £51732. 17*s.* 2¾*d.* × 5
(11) 301620 × 5
(12) £23154. 8*s.* 8¾*d.* × 5
(13) 510847 × 5
(14) £34088. 13*s.* 9½*d.* × 5
(15) 639756 × 5
(16) £62977. 5*s.* 3¼*d.* × 5
(17) 847538 × 5
(18) £79863. 19*s.* 10*d.* × 5
(19) 925317 × 5
(20) £18641. 7*s.* 11¼*d.* × 5

10. Learn by heart:

1 × 6 = 6 (read, once six is six).
2 × 6 = 12 (twice six is twelve).
3 × 6 = 18 (three times six is eighteen).
4 × 6 = 24
5 × 6 = 30
6 × 6 = 36
7 × 6 = 42
8 × 6 = 48
9 × 6 = 54
10 × 6 = 60

Also:

6 × 1 = 6 (read, six times one is six).
6 × 2 = 12 (six times two is twelve).
6 × 3 = 18 (six times three is eighteen).
6 × 4 = 24
6 × 5 = 30
6 × 6 = 36
6 × 7 = 42
6 × 8 = 48
6 × 9 = 54
6 × 10 = 60

Notice that the sum of the digits in the successive answers follows the order 6, 3, 9.

Exercise XI. (*e*).

(1) How much money in 6 bags of £4807. 6*s.* 11¼*d.* each?

(2) Find the number of pages in 6 volumes, if each volume has 483 pages.

(3) Find the cost of 6 locomotives, if each costs £2095. 13*s.* 4*d.*

(4) Find the length of the sides of a hexagon, if each is 529 inches in length.

(5) The daily expenditure of an office in the city is £17. 13*s.* 10*d.* How much is this a-week?

(6) A mile has 880 fathoms. How many feet has it (a fathom being 6 feet)?

(7) 416296 × 6
(8) £40529. 6s. 0½d. × 6
(9) 278409 × 6
(10) £51732. 17s. 2¾d. × 6
(11) 301620 × 6
(12) £23154. 8s. 8¾d. × 6
(13) 510847 × 6
(14) £34088. 13s. 9½d. × 6
(15) 639756 × 6
(16) £62977. 5s. 3¼d. × 6
(17) 847538 × 6
(18) £79863. 19s. 10d. × 6
(19) 925317 × 6
(20) £18641. 7s. 11¼d. × 6

11. Learn by heart:

1 × 7 = 7 (read, once seven is seven).
2 × 7 = 14 (twice seven is fourteen).
3 × 7 = 21 (three times seven is twenty-one)
4 × 7 = 28
5 × 7 = 35
6 × 7 = 42
7 × 7 = 49
8 × 7 = 56
9 × 7 = 63
10 × 7 = 70

Also:

7 × 1 = 7 (read, seven times one is seven)
7 × 2 = 14 (seven times two is fourteen).
7 × 3 = 21 (seven times three is twenty-one).
7 × 4 = 28
7 × 5 = 35
7 × 6 = 42
7 × 7 = 49
7 × 8 = 56
7 × 9 = 63
7 × 10 = 70

EXERCISE XI. (*f*).

(1) How much money in 7 bags, each containing £6429. 15s. 3½d.?

(2) How many minutes in a week, a day consisting of 1440 minutes?

(3) If living costs me 13s. 9½d. a-day, how much do I require a-week?

(4) Find the length of the sides of a heptagon, each side being 748 inches long.

(5) Find the difference between 52 weeks and a year (of 365 days).

(6) The seventh part of a ton is 320 lbs. How many lbs. are there in a ton?

(7) 416296 × 7
(8) £40529. 6s. 0½d. × 7
(9) 278409 × 7
(10) £51732. 17s. 2¾d × 7
(11) 301620 × 7
(12) £23154. 8s. 8¾d. × 7
(13) 510847 × 7
(14) £34088. 13s. 9½d. × 7
(15) 639756 × 7
(16) £62977. 5s. 3¼d. × 7
(17) 847538 × 7
(18) £79863. 19s. 10d. × 7
(19) 925317 × 7
(20) £18641. 7s. 11¼d. × 7

12. Learn by heart:

$1 \times 8 = 8$ (read, once eight is eight).
$2 \times 8 = 16$ (twice eight is sixteen).
$3 \times 8 = 24$ (three times eight is twenty-four)
$4 \times 8 = 32$
$5 \times 8 = 40$
$6 \times 8 = 48$
$7 \times 8 = 56$
$8 \times 8 = 64$
$9 \times 8 = 72$
$10 \times 8 = 80$

Also:

$8 \times 1 = 8$ (read, eight times one is eight)
$8 \times 2 = 16$ (eight times two is sixteen).
$8 \times 3 = 24$ (eight times three is twenty-four).
$8 \times 4 = 32$
$8 \times 5 = 40$
$8 \times 6 = 48$
$8 \times 7 = 56$
$8 \times 8 = 64$
$8 \times 9 = 72$
$8 \times 10 = 80$

EXERCISE XI. (*g*).

(1) How much money in 8 bags, each containing £5786. 13*s.* 7½*d.*?

(2) A mile has 8 furlongs, and a furlong has 660 feet. How many feet in a mile?

(3) If one man pays 3*s.* 9¼*d.* for his dinner, what will a party of 8 pay?

(4) Find the length of the sides of an octagon, if each side is 237 inches long.

(5) Find the cost of 8 railway tickets at £1. 15*s.* 6*d.* each.

(6) A certain city has 47,968 houses; if, on an average, each has 8 windows, how many windows are there altogether?

(7) 416296×8
(8) £40529. 6*s.* 0½*d.* × 8
(9) 278409×8
(10) £51732. 17*s.* 2¾*d.* × 8
(11) 301620×8
(12) £23154. 8*s.* 8¾*d.* × 8
(13) 510847×8
(14) £34088. 13*s.* 9½*d.* × 8
(15) 639756×8
(16) £62977. 5*s.* 3¼*d.* × 8
(17) 847538×8
(18) £79863. 19*s.* 10*d.* × 8
(19) 925317×8
(20) £18641. 7*s.* 11¼*d.* × 8

13. Learn by heart:

$1 \times 9 = 9$ (read, once nine is nine).
$2 \times 9 = 18$ (twice nine is eighteen).
$3 \times 9 = 27$ (three times nine is twenty-seven)
$4 \times 9 = 36$
$5 \times 9 = 45$
$6 \times 9 = 54$
$7 \times 9 = 63$
$8 \times 9 = 72$
$9 \times 9 = 81$
$10 \times 9 = 90$

Also:

$9 \times 1 = 9$ (read, nine times one is nine).
$9 \times 2 = 18$ (nine times two is eighteen).
$9 \times 3 = 27$ (nine times three is twenty-seven).
$9 \times 4 = 36$
$9 \times 5 = 45$
$9 \times 6 = 54$
$9 \times 7 = 63$
$9 \times 8 = 72$
$9 \times 9 = 81$
$9 \times 10 = 90$

Notice that the sum of the digits is *always* 9; moreover, that the figure in the tens' place is always one less than the multiplier. Thus in 6 × 9, the tens' figure is 5, and the units' figure the difference between 5 and 9.

EXERCISE XI. (*h*).

(1) How much money in 9 bags, each containing £5437. 6*s*. 7¼*d*.?

(2) How many ninepins are there in 403 sets?

(3) What salary shall I draw in 9 months at £13. 2*s*. 6*d*. a-month?

(4) Find the length of the sides of a nonagon, each side being 222 inches long.

(5) What will be the cost of a terrace of 9 houses at £1166. 13*s*. 4*d*. each?

(6) A sovereign weighs 123 grains, how many grains should 9 sovereigns weigh?

(7) 416296 × 9
(8) £40529. 6*s*. 0½*d*. × 9
(9) 278409 × 9
(10) £51732. 17*s*. 2¾*d*. × 9
(11) 301620 × 9
(12) £23154. 8*s*. 8¾*d*. × 9
(13) 510847 × 9
(14) £34088. 13*s*. 9½*d*. × 9
(15) 639756 × 9
(16) £62977. 5*s*. 3¼*d*. × 9
(17) 847538 × 9
(18) £79863. 19*s*. 10*d*. × 9
(19) 925317 × 9
(20) £18641. 7*s*. 11¼*d*. × 9

14. Learn by heart:

1 × 10 = 10 (read, once ten is ten).	10 × 1 = 10 (read, ten times one is ten).
2 × 10 = 20 (twice ten is twenty).	10 × 2 = 20 (ten times two is twenty).
3 × 10 = 30 (three times ten is thirty.	10 × 3 = 30 (ten times three is thirty).
4 × 10 = 40	10 × 4 = 40
5 × 10 = 50	10 × 5 = 50
6 × 10 = 60	10 × 6 = 60
7 × 10 = 70	10 × 7 = 70
8 × 10 = 80	10 × 8 = 80
9 × 10 = 90	10 × 9 = 90
10 × 10 = 100	10 × 10 = 100

(The right column is headed "Also:")

Multiply 68437 by 10.

$$\begin{array}{r} 68437 \\ 10 \\ \hline 684370 \end{array}$$

Wording: 70, 0′, carry 7; 30, 37′, carry 3; 40, 43′, carry 4; 80, 84′, carry 8; 60, 68′.

Similarly $51376 \times 10 = 513760$, and $123456 \times 10 = 1234560$. Notice that the figures in the product are the same as those in the multiplicand, with the addition of a cipher in the units' place. The question arises, Will this always be so? On trying any number of cases, the coincidence will be found to hold. If we were to trust to experience only, which is called reasoning by analogy, our conclusions might be correct, but would not carry to our minds the full conviction that is forced upon us by logical deduction. In many cases of reasoning we are obliged to rely upon mere analogy; but in this case we can prove that the coincidence not only will, but must, always hold.

Compare the multiplicand and product in the above example:

XM.	M.	C.	X.	I.		CM.	XM.	M.	C.	X.	I.
6	8	4	3	7	and	6	8	4	3	7	0.

The 7 in the former stood for		7 units,	in the latter for		7 tens.
3	,,	3 tens,		,,	3 hundreds.
4	,,	4 hundreds,		,,	4 thousands.
8	,,	8 thousands,		,,	8 ten-thousands.
6	,,	6 ten-thousands,		,,	6 hundred-thousands.

Thus we see that, by putting the cipher in the units' place, each figure in the multiplicand has been moved one place higher in the numeration scale, and has therefore been made ten times as valuable, or has been *multiplied by* 10, and multiplying each figure by 10 is equivalent to multiplying the whole by 10.

Learn by heart: *To multiply any number by* 10, *put on a cipher in the units' place, and copy the remaining figures.*

Q. Apply this rule to the following: £369. 17*s.* $8\frac{3}{4}$*d.* $\times$ 10.
A. £369. 17*s.* $8\frac{3}{4}0$*d.*

Teacher. It is evident that the rule as it stands does not apply here, and for this reason: We are dealing with different numeration scales in the same sum, and the cipher as placed above does not alter the positions of the different figures in their respective scales. There being four different scales, four ciphers will be required, thus: £3690. 170*s.* $80\frac{30}{4}$*d.* This product, though not inaccurate, is evidently inconvenient, and we therefore proceed as follows:

£.	s.	d.
369	17	$8\frac{3}{4}$
		10
3698	17	$3\frac{1}{2}$

Wording: 30, $\frac{1}{2}$'$d.$, carry 7; 80, 87, 3', carry 7; 7', 17 (half-sovereigns), 1', carry 8; 8', 9', 6', 3'. To multiply shillings by 10, we have only to call them half-sovereigns. The pounds need not be worked, and we have only to put the figure carried from the half-sovereigns in the place of the cipher that is added in multiplying the pounds by 10.

Exercise XI. (*i*)

(1) How much money in 10 bags, each containing £5846. 17*s.* $10\frac{3}{4}d.$?

(2) How many fingers (and thumbs) will 798 men have?

(3) Find the value of 10 shares at £93. 17*s.* 6*d.* each.

(4) How many hundreds in 5786 thousands?

(5) Find the expenditure of 10 years at £1048. 19*s.* 10*d.* a-year.

(6) How many tens in 75964 hundreds?

(7) 416296 × 10

(8) £40529. 6*s.* $0\frac{1}{4}d.$ × 10

(9) 278409 × 10

(10) £51732. 17*s.* $2\frac{3}{4}d.$ × 10

(11) 301620 × 10

(12) £23154. 8*s.* $8\frac{3}{4}d.$ × 10

(13) 501847 × 10

(14) £34088. 13*s.* $9\frac{1}{4}d.$ × 10

(15) 639756 × 10

(16) £62977. 5*s.* $3\frac{1}{4}d.$ × 10

(17) 847538 × 10

(18) £79863. 19*s.* 10*d.* × 10

(19) 925317 × 10

(20) £18641. 7*s.* $11\frac{1}{4}d.$ × 10

15. The interpretation of the symbol ×, even as extended in § 4, is inapplicable to such a case as £5. 3*s.* 10*d.* × 1, since in the symbol 1 the notion of *repetition* does not enter. The interpretation must therefore be still further extended; but every extension of the interpretation of symbols in Arithmetic and throughout Mathematics must fulfil the two following conditions:

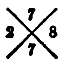
(*a*) It must give an intelligible meaning to the symbol in the new case which requires this extension.

(*b*) The new wording must not alter the sense attached to the symbol in the earlier cases, and must remain subject to the general rules already established. For example, whatever extension of meaning we may in future have occasion to give to the symbol +, it must always be true that the sum of two or more numbers will be

the same, in whatever order the addenda be taken; or whatever meaning we may in future have occasion to give to the symbol $\times$, it must always be true that the product of two numbers will be the same in whatever order they are taken.

Learn by heart: *This sign* ($\times$) *indicates that the number on either side of it is to be* TAKEN *as many* TIMES *as is indicated by the number on the other side of it.*

16. Test of accuracy by casting out nines. 32645×8.



$$\begin{array}{r} 32645 \\ 8 \\ \hline 261160 \end{array}$$

Cast out nines from the multiplier and multiplicand and write the results (8 and 2) in the spaces right and left of a cross, as in the margin; multiply these results (16), cast out nines, and place the result (7) in the upper space; lastly, cast out nines from the product found, and write the result in the lower space. If the figures in the upper and lower spaces do not agree, there must be an error in the working.

EXERCISE XII. (*a*).

(1) Find the cost of 7 articles, at £3. 4*s.* 5*d.* each.

(2) Multiply £68419. 11*s.* $2\frac{1}{2}$*d.* by 1.

(3) Multiply £768. 13*s.* 9*d.* by 10.

(4) Multiply 7686875 by 10.

(5) Repeat £9. 13*s.* $8\frac{1}{4}$*d.* 6 times.

(6) Simplify $9684375 + 9684375 + 9684375 + 9684375$.

(7) Take £19. 14*s.* $2\frac{3}{4}$*d.* 7 times.

(8) Find the amount of 7 collections, each containing 197114583 things.

(9) What shall I spend in 5 years if I spend £476. 13*s.* 8*d.* a-year?

(10) How many men are there in 6 regiments of 936 men each?

(11) How many sheep in 8 flocks of 147 sheep each?

(12) How much must be paid for 6 locomotives at £3126 each?

(13) How far shall I travel in 9 days, if I travel 672 miles a-day?

(14) Find the double of £5943. 14s. 6d.

(15) Find the length of 4 journeys of 1768 miles each.

(16) The Bank of England sent to the Bank of France 3 chests of money, each containing £147,916. How much money in all?

(17) How many legs have 683 sheep, 429 oxen, and 715 horses?

(18) How many legs have 326 men, 519 ostriches, and 478 canaries?

(19) How many toes have they?

(20) I bought 9 articles at 13*s.* 10½*d.* each; 6 of them I sold at 15*s.* 4½*d.* each, and 3 at £1 each. How much did they all cost me, for how much did I sell them all, and what profit did I make?

Exercise XII. (*b*).

(1) To 7 times 4709, add 7037.

(2) To 4 times 6835, add 5 times 216.

(3) Find the difference between 6 times 5987 and 3 times 9412.

(4) Which is the greater, and by how much, 8×3104, or 7×4080?

(5) Multiply the sum of £8417. 13*s.* 11*d.*, £359. 16*s.* 8*d.*, £1043. 3*s.* 2½*d.*, £6. 11*s.* 10¼*d.*, £428. 5*s.* 9½*d.*, and £7932. 17*s.* 11*d.*, by 9.

(6) Multiply the difference between £10,000 and £8342. 17*s.* 10¼*d* by 10.

(7) Multiply the difference between 6×2304 and 8×1728 by 10.

(8) Multiply the sum of 18451 and 18444 by their difference.

(9) Prove that 1 × £3267. 12*s.* 10*d.* + 2 × £3267. 12*s.* 10*d.* + 3 × £3267. 12*s.* 10*d.* + 4 × £3267. 12*s.* 10*d.* + 5 × £3267. 12*s.* 10*d.* + 6 × £3267. 12*s.* 10*d.* + 7 × £3267. 12*s.* 10*d.* + 8 × £3267. 12*s.* 10*d.* + 9 × £3267. 12*s.* 10*d.* = 9 × £3267. 12*s.* 10*d.* taken 5 times.

(10) Prove that the same is true of 105894.

(11) Also of £1084. 7*s.* 11¼*d.*

(12) Also of 67321.

(13) Also of £4729. 18*s*. 3*d*.

(14) Also of 568139.

(15) Also of £65,478. 17*s*. $9\frac{3}{4}$*d*.

17. Let us compare the meanings of the three symbols +, —, ×, in accordance with the interpretations given in Ch. III. § 2, Ch. IV § 2, and Ch. V. § 15.

7 chairs + 3 chairs; this means that 7 chairs and 3 chairs are to be added together. *Ans.* 10 chairs.

Again, 7 chairs — 3 chairs; this means that 3 chairs are to be taken away from 7 chairs. *Ans.* 4 chairs.

Again, 7 chairs × 3 chairs; this should mean either that 7 chairs are to be taken 3 chairs' times, or that 3 chairs are to be taken 7 chairs' times; but what sense can we attach to either of these expressions? Evidently none whatever. Let us then investigate a still earlier case.

1 chair + 1 chair? *Ans.* 2 chairs.

1 chair — 1 chair? *Ans.* 0 chairs.

1 chair × 1 chair? This also is perfectly, and perhaps more obviously, meaningless.

Let us therefore consider the first introduction of the symbol × in § 1 of this Chapter. £7 + £7 + £7 + £7 + £7 + £7 has been contracted into £7 × 6, or 6 × £7; similarly, 7 chairs + 7 chairs + 7 chairs + 7 chairs + 7 chairs + 7 chairs is contracted into 6 × 7 chairs, or 7 chairs × 6. In every case, then, the 6 has one and the same meaning, viz. 6 *times*, and is perfectly independent of the nature of the thing represented by the 7. Hence such a problem as £7. 8*s*. 10*d*. × £3. 4*s*. $11\frac{1}{4}$*d*. is sheer nonsense, and admits of no solution.*

Learn by heart: *In Addition and Subtraction we must always have the* SAME *kind of units, viz. so many things of one kind added to or taken from so many things of the* SAME *kind; but in Multiplication*

* In Algebra, when the first problem of multiplication presents itself, a still further extension of the meaning of the symbol × becomes necessary; but even then there will be insuperable difficulties in attaching a *definite* meaning to this problem.

we must always have TWO DIFFERENT *kinds of units, viz. so many* THINGS *repeated or taken so many* TIMES.

18. Apply these considerations to the following:

£8 + £0. *Ans.* £8.

£0 + £8. *Ans.* £8.

£0 × 8. *Ans.* £0.

£8 × 0. This *new* case requires investigation. It cannot mean that £8 are not to be multiplied at all, but only left alone, for then it would be equivalent to £8 × 1; the only sense attachable to the expression is that £8 are to be taken *no* times, and therefore that £8 × 0 = £0.*

Or again: We have seen that 7 times 6 = 6 times 7 (§ 4); therefore 0 times 8 should = 8 times 0, as the above reasoning shews to be true, and this strengthens our conviction that we have reasoned correctly.

£8 — £0. *Ans.* £8.

£0 — £8. This has no meaning.

Learn by heart: *In Addition and Multiplication the numbers may be taken in any order we please, but not so in Subtraction.*

CHAPTER VI.

MULTIPLICATION—*continued.*

1. It is required to multiply £587. 13*s.* 11*d.* by 13. As our knowledge of the Multiplication Table only extends to 10 times, some artifice must be adopted by which this defect can be supplied.

* "There is a number of boxes, none of which contain anything. How much do all together contain?

"If a be the number of boxes, then 0 repeated a times, or $a \times 0$, is 0.

"There is a box full of gold, of which no part whatsoever belongs to A. How much belongs to A?

"If p be the number of pounds of gold in the box, then A's part is $0 \times p$, or 0."
(*De Morgan's Algebra*, Introduction.)

Here (×) has the fuller meaning, which we shall give hereafter. See Book II. *Ch. III. § 3.*

$13 = 10 + 3 = 9 + 4 = 8 + 5 = 7 + 6$, $\therefore$ to repeat any quantity 13 times we may repeat it first 10 and then 3 times, or else first 9 and then 4 times, or 8 and 5 times, or 7 and 6 times, and the sum of each pair of products must be the same.

Thus a person who cannot carry more than 10 boxes of a given weight, can remove 13 boxes from one place to another *in the smallest possible number of journeys*, by dividing his work in any one of the four ways given above. Applying this principle to the problem proposed, we may proceed as follows:

1st mode.

	£	s.	d.
A	£587	13	11
		13 = 10 + 3	
B	5876	19	2
C	1763	1	9
D	7640	0	11

The line marked D consists of the sum of the lines B and C, $\therefore$ $D = B + C$; again, B was obtained by taking A 10 times, $\therefore$ $B = 10 \times A$; similarly, $C = 3 \times A$, $\therefore$ $B + C = 10 \times A + 3 \times A$, or $13 \times A$; but $B + C = D$, $\therefore$ $D = 13 \times A$.

2nd mode.

	£	s.	d.
A	£587	13	11
		13 = 9 + 4	
B	5289	5	3
C	2350	15	8
D	7640	0	11

$D = B + C = 9 \times A + 4 \times A = 13 \times A.$

3rd mode.

£587	13	11
	13 = 8 + 5	
4701	11	4
2938	9	7
7640	0	11

4th mode.

£587	13	11
	13 = 7 + 6	
4113	17	5
3526	3	6
7640	0	11

Multiply 87495 by 13 in 4 different ways, shewing that the results coincide.

87495	87495	87495	87495
13 = 10 + 3	13 = 9 + 4	13 = 8 + 5	13 = 7 + 6
874950	787455	699960	612465
262485	349980	437475	524970
1137435	1137435	1137435	1137435

Inspection of the above four methods of multiplication of an abstract number by 13, shews that the selection of $10 + 3$ is the easiest, because multiplication by 10 involves no labour. This is therefore the method commonly chosen for abstract numbers, and will in practice be found the most convenient in money sums also.

Multiplication by 11 and by 12 may be done in one line, because the table for 11 is learnt by one inspection, and that for 12 has already been, incidentally, committed to memory.

$1 \times 11 = 11$	$1 \times 12 = 12$
$2 \times 11 = 22$	$2 \times 12 = 24$
$3 \times 11 = 33$	$3 \times 12 = 36$
$4 \times 11 = 44$	$4 \times 12 = 48$
$5 \times 11 = 55$	$5 \times 12 = 60$
$6 \times 11 = 66$	$6 \times 12 = 72$
$7 \times 11 = 77$	$7 \times 12 = 84$
$8 \times 11 = 88$	$8 \times 12 = 96$
$9 \times 11 = 99$	$9 \times 12 = 108$
$10 \times 11 = 110$	$10 \times 12 = 120$

Exercise XIII.

(1) Find all the possible pairs of numbers (none exceeding 10) whose sum is 14.

(2) Do the same with 15, 16, 17, 18, 19.

(3) Multiply £4279. 13*s.* $8\frac{1}{2}$*d.* by 14, in four different ways, shewing that the results coincide.

(4) Multiply 2756983 by 15, in three different ways.

(5) Multiply £835. 11*s.* $10\frac{3}{4}$*d.* by 16, in three different ways.

(6) Multiply 4102568 by 17, in two different ways.

(7) Multiply £6179. 14*s.* $9\frac{1}{4}$*d.* by 18, in two different ways.

(8) Multiply 3490716 by 19.

(9) Multiply £674. 15*s.* 9*d.* by 11.

(10) Multiply 6747875 by 11.

(11) Multiply £18792. 9*s.* $7\frac{1}{4}$*d.* by 12.

(12) Multiply 1356794 by 12.

2. Multiply 478916 by 20.

A	478916
	$20 = 10 + 10$
B	4789160
C	4789160
D	9578320

The lines B and C being equal, it is evident that, instead of copying line B, we might have multiplied it by 2 (see Ch. V. § 1), thus:

$$\begin{array}{r} 478916 \\ \underline{20 = 10+10 = 10\times 2} \\ 4789160 \\ \underline{2} \\ 9578320 \end{array}$$

Similarly 478916 × 30:

1st mode.

$$\begin{array}{r} 478916 \\ \underline{30 = 10+10+10} \\ 4789160 \\ 4789160 \\ \underline{4789160} \\ 14367480 \end{array}$$

2nd mode.

$$\begin{array}{lr} A & 478916 \\ & \underline{30 = 10\times 3} \\ B & 4789160 \\ & \underline{3} \\ & 14367480 \end{array}$$

478916 × 13:

$$\begin{array}{lr} A & 478916 \\ & \underline{13 = 10+3} \\ B & 4789160 \\ C & \underline{1436748} \\ D & 6225908 \end{array}$$

Compare the multiplication by 30 with the multiplication by 13. In the first, the line B, or the *ten* line, is multiplied by 3, and this *at once* yields the answer. In the second, it is not line B, but line A, that is multiplied by 3, and this multiplication only yields us line C, which has yet to be added to line B to give the answer, line D. In other words, 30 times is 3 *times* 10 times; 13 times is 10 times *and* 3 times put together.

The above process of multiplication by 30 may be contracted, as the line B is merely a copy of A with the addition of a 0 in the units' place; we may therefore obtain the result in one line by writing down the 0 and multiplying by 3 at once, thus:

$$\begin{array}{r} 478916 \\ \underline{30} \\ 14367480 \end{array}$$

Similarly, we multiply by 40 by writing a 0 in the units' place

and multiplying by 4; and so on for 50, 60, 70, 80, 90, where we write a 0 and multiply by 5, 6, 7, 8, 9, respectively.

Exercise XIV. (*a*).

(1) 3469875 × 20
(2) 783125 × 30
(3) 1874167 × 40
(4) 1356875 × 50
(5) 96438125 × 60
(6) 27649583 × 70
(7) 78415625 × 80
(8) 753125 × 90

In multiplication of money, line B (see below) is not merely a copy of line A with the addition of a 0, and this contraction is therefore inapplicable.

£324. 8*s.* 7*d.* × 50:

A	£324	8	7
			10
B	3244	5	10
			5
C	16221	9	2

Exercise XIV. (*b*).

(1) £346. 19*s.* 9*d.* × 20
(2) £78. 6*s.* 3*d.* × 30
(3) £187. 8*s.* 4*d.* × 40
(4) £13. 11*s.* 4½*d.* × 50
(5) £964. 7*s.* 7½*d.* × 60
(6) £276. 9*s.* 11*d.* × 70
(7) £78. 8*s.* 3¾*d.* × 80
(8) 15*s.* 0¾*d.* × 90

Exercise XIV. (*c*).

(1) £29. 13*s.* 8¼*d.* × 30
(2) £29. 13*s.* 8¼*d.* × 13
(3) 29684375 × 30
(4) 29684375 × 13
(5) £645. 15*s.* 8*d.* × 14
(6) £645. 15*s.* 8*d.* × 40
(7) 6457833 × 40
(8) 6457833 × 14
(9) £10. 16*s.* 9¾*d.* × 50
(10) £10. 16*s.* 9¾*d.* × 15
(11) 10840625 × 50
(12) 10840625 × 15
(13) 9*s.* 10½*d.* × 16
(14) 9*s.* 10½*d.* × 60
(15) 49375 × 16
(16) 49375 × 60
(17) 29364583 × 17
(18) £29. 7*s.* 3½*d.* × 70
(19) £29. 7*s.* 3½*d.* × 17
(20) 29364583 × 70
(21) £7. 18*s.* 5¾*d.* × 18
(22) £7. 18*s.* 5¾*d.* × 80
(23) 79239583 × 18
(24) 79239583 × 80
(25) £56. 19*s.* 11¾*d.* × 19
(26) £56. 19*s.* 11¾*d.* × 90
(27) 569989583 × 19
(28) 569989583 × 90

(29) £58. 17s. 8¼d. × 15
(30) £58. 17s. 8¼d. × 50
(31) 58884375 × 15
(32) 58884375 × 50
(33) £194. 17s. 2d. × 17
(34) £194. 17s. 2d. × 70
(35) 19485833 × 17
(36) 19485833 × 70
(37) £962. 7s. 6d. × 19
(38) £962. 7s. 6d. × 90
(39) 962375 × 19
(40) 962375 × 90

3. Multiply 2628417 by 53.

```
  2628417
  53 = 50+3
---------
131420850
  7885251
---------
139306101
```

Multiply £36. 9s. 8½d. by 67.

A	£36	9	8½	× 67; 67 = 60 + 7
			10	
B	364	17	1	Line B = 10 × A
			6	,, C = 6 × B = 60 × A
C	2189	2	6	,, D = 7 × A
D	255	7	11½	∴ C + D or E = 60 × A + 7 × A = 67 × A
E	2444	10	5½	

Exercise XV.

(1) £4. 16s. 7½d. × 23
(2) £4. 16s. 7½d. × 32
(3) 483125 × 23
(4) 483125 × 32
(5) £12. 13s. 5¾d. × 34
(6) £12. 13s. 5¾d. × 43
(7) 126739583 × 34
(8) 126739583 × 43
(9) £26. 17s. 11d. × 46
(10) £26. 17s. 11d. × 64
(11) 2689583 × 46
(12) 2689583 × 64
(13) £34. 14s. 3¾d. × 57
(14) £34. 14s. 3¾d. × 75
(15) 34715625 × 57
(16) 34715625 × 75
(17) £5. 7s. 6¼d. × 61
(18) £5. 7s. 6¼d. × 16
(19) 53760416 × 61
(20) 53760416 × 16
(21) £27. 9s. 8¾d. × 77
(22) 274864583 × 77
(23) £6. 19s. 10¼d. × 82
(24) £6. 19s. 10¼d. × 28
(25) 69927083 × 82
(26) 69927083 × 28
(27) £39. 5s. 7½d. × 98
(28) £39. 5s. 7½d. × 89
(29) 3928125 × 98
(30) 3928125 × 89

4. In some cases this process of multiplication may be somewhat contracted, e.g. $56 = 50 + 6$, but it also equals $8 + 8 + 8 + 8 + 8 + 8 + 8 = 7$ times 8, or $7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 = 8$ times 7.

Multiply 78429 by 56.

1st mode.	2nd mode.	3rd mode.
78429 A	78429 A	78429 A
$56=50+6$	$56=7\times 8$	$56=8\times 7$
3921450 B	627432 E	549003 F
470574 C	7	8
4392024 D	4392024 D	4392024 D

$D = B + C$; but $B = 50 \times A$, and $C = 6 \times A$, $\therefore B + C = 56 \times A$.
Again, $D = 7 \times E$; but each $E = 8 \times A$, $\therefore D = 7 \times 8 \times A = 56 \times A$.
Again, $D = 8 \times F$; but each $F = 7 \times A$, $\therefore D = 8 \times 7 \times A = 56 \times A$.

Multiply £478. 5*s*. 7*d*. by 56.

1st mode.			2nd mode.			3rd mode.		
£478	5	7	£478	5	7	£478	5	7
		10			8			7
4782	15	10	3826	4	8	3347	19	1
		5			7			8
23913	19	2	26783	12	8	26783	12	8
2869	13	6						
26783	12	8						

Exercise XVI.

(1) Multiply £16. 15*s*. $8\frac{1}{4}d$. by 56 in three different ways, shewing that the results coincide.

(2) 16784375×56 in three ways.

(3) £82. 9*s*. $3\frac{1}{2}d$. $\times$ 45 in three ways.

(4) 82464583×45 in three ways.

(5) £143. 6*s*. $10\frac{3}{4}d$. $\times$ 20 in four ways.

(6) 15809×20 in three ways.

(7) State the different ways in which we can multiply by 24, 30, 36, 42, 60, 63.

5. Continued Product.

When there are several numbers, and the first is multiplied by the second, the product by the third, that product by the fourth, and so on, the last product is called the Continued Product of all the numbers.

The continued product of 7, 8, 9, 11, is $7 \times 8 \times 9 \times 11 = 5544$.

The factors 7, 8, 9, 11, may be taken in any order; for any two of them, say 8 and 9, may change places, because $8 \times 9 = 9 \times 8$ (Ch. V. § 4).

Thus $7 \times 8 \times 9 \times 11 = 7 \times 9 \times 8 \times 11$.

In this way, from $7 \times 8 \times 9 \times 11$, any order, say $8 \times 11 \times 9 \times 7$, can be obtained by the following steps:

$7 \times 8 \times 9 \times 11 = 8 \times 7 \times 9 \times 11 = 8 \times 9 \times 7 \times 11 = 8 \times 9 \times 11 \times 7$
$= 8 \times 11 \times 9 \times 7$.

Illustration: $4 \times 3 \times 5$ may be represented thus—

```
. . . .   . . . .   . . . .   . . . .   . . . .
. . . .   . . . .   . . . .   . . . .   . . . .
. . . .   . . . .   . . . .   . . . .   . . . .
```

which can be seen to be $(3 \times 4) \times 5$, or $(4 \times 3) \times 5$, or $(5 \times 4) \times 3$.

If each dot were supposed to be replaced by a collection, say of 7, we should see that $7 \times 3 \times 4 \times 5 = 7 \times 4 \times 5 \times 3$, &c.

6. Multiply 78426 by 100. $100 = 10 \times 10$, hence we may proceed as follows:

$$\begin{array}{r} 78426 \\ 10 \\ \hline 784260 \\ 10 \\ \hline 7842600 \end{array}$$

But on comparing the product with the multiplicand, we find that we have added two ciphers on the right-hand side. In fact, by the addition of these two ciphers, each digit of the multiplicand has been removed two places higher in the numeration scale, and has thereby had its value raised a hundredfold. Similarly, a number is multiplied by 1000 by adding 3 ciphers on the right, by 10,000 by adding 4 ciphers, and so on.

The numbers 10, 100, 1000, &c., which are 10, 10×10, $10 \times 10 \times 10$, &c., are called Powers of 10.

Learn by heart: *To multiply a number by any power of* 10, *put on as many ciphers to the right of the multiplicand as there are ciphers in the multiplier.* This rule is of course inapplicable to multiplication of money. (Ch. V. p. 53.)

Multiply £43. 7*s.* $8\frac{1}{2}$*d.* by 10000.

	£	s.	d.	
A	£43	7	$8\frac{1}{2}$	
			10	
B	433	17	1	Line B = 10 × A
			10	,, C = 10 × B = 100 × A
C	4338	10	10	,, D = 10 × C = 100 × B = 1000 × A
			10	,, E = 10 × D = 100 × C = 1000 × B = 10000 × A
D	43385	8	4	
			10	
E	433854	3	4	

Exercise XVII.

(1) £529. 17*s.* 8*d.* × 100.

(2) £8342. 5*s.* $9\frac{1}{2}$*d.* × 1000.

(3) £7. 11*s.* $2\frac{3}{4}$*d.* × 10000.

(4) £845. 4*s.* 10*d.* × 100.

(5) £3410 × 1000000.

(6) 3*s.* $2\frac{1}{2}$*d.* × 100000.

(7) Find the value of a million penny postage stamps.

(8) If one rupee is worth 1*s.* $11\frac{1}{4}$*d.*, what is the value of a lac of rupees (100,000)?

(9) 42748 × 100.

(10) 609 × 1000.

(11) 5040 × 10000.

(12) 170000 × 100000.

(13) 19 × 1000000.

(14) 1 × 1000.

(15) 3000 × 1000.

(16) 1000 × 100.

(17) 100 × 10000.

(18) 1000 × 1000.

(19) 10000 × 10000.

(20) A million × a million.

7. Multiply 5267 by 400. $400 = 4 \times 100$; we therefore multiply by 100 and then this product by 4, thus:

5267	
100	or in one line, as in § 2:
526700	5267
4	400
2106800	2106800

Multiply £83. 4*s.* $10\frac{1}{4}$*d.* by 600.

£	s.	d.
£83	4	$10\frac{1}{4}$
		10
832	8	$6\frac{1}{2}$
		10
8324	5	5
		6
49945	12	6

Multiply 78652 by 347. $347 = 300 + 40 + 7$; we therefore take the multiplicand first 300 times, then 40 times, and lastly 7 times, and add these three products together.

78652		
347		
23595600	=	300 times
3146080	=	40 times
550564	=	7 times
27292244	=	347 times

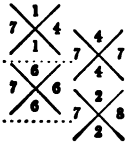
Multiply 43756 by 4768.

43756		
4768		
175024000	=	4000 times
30629200	=	700 times
2625360	=	60 times
350048	=	8 times
208628608	=	4768 times

N.B. The test by casting out nines will not detect an error in the order or position of the figures, since the sum of the digits would be the same whatever their arrangement in the multiplier, the multiplicand or any of the products.

Supposing, however, an error to be indicated by the test, it is not necessary to go through the whole work again, as each line can be tested separately: e.g.

For the 4000 line the scheme is.........
,, 700 ,, ...
,, 60 ,, ...
,, 8 ,, ...



Multiply 75948 by 4060.

```
     75948
      4060
 303792000 = 4000 times
   4556880 =   60 times
 308348880 = 4060 times
```

13467 × 10110.

```
     13467
     10110
 134670000 = 10000 times
   1346700 =   100 times
    134670 =    10 times
 136151370 = 10110 times
```

Multiply £15. 7*s*. 8¾*d*. by 347.

```
A  £15  7  8¾×7
          10
B  153 17  3½×4
          10
C 1538 12 11
           8
  4615 18  9 =3×C=3×100 times=300 times
   615  9  2 =4×B=4× 10 times= 40 times
   107 14  1¼=7×A=7×  1 time =  7 times
  5339  2  0¼                 347 times
```

£6. 15*s*. 10½*d*. × 8040.

```
A    £6 15 10½
          10
B    67 18  9×4
          10
C   679  7  6
          10
D  6793 15  0
            8
  54350  0  0=8×D=8×1000 times=8000 times
    271 15  0=4×B=4×  10 times=  40 times
  54621 15  0                  8040 times
```

Multiply £3. 7s. 8½d. by 2400. $2400 = 2000 + 400 = 3 \times 8 \times 100 = 4 \times 6 \times 100 = 2 \times 12 \times 100 = 8 \times 3 \times 100 = 8 \times 100 \times 3 = 3 \times 100 \times 8$, &c. &c.

1st mode.

3	7	8½	
		10	
33	17	1	
		10	
338	10	10	×4
		10	
3385	8	4	
		2	
6770	16	8	=2000 times
1354	3	4	= 400 times
8125	0	0	=2400 times

2nd mode.

3	7	8½	
		10	
33	17	1	
		10	
338	10	10	=100 times
		8	
2708	6	8	=800 times
		3	
8125	0	0	$=3 \times 800$ times$=2400$ times

3rd mode.

£3	7	8½	
		8	
27	1	8	=8 times
		3	
81	5	0	$=3 \times 8$ times$=24$ times
		10	
812	10	0	$=10 \times 24$ times$=240$ times
		10	
8125	0	0	$=10 \times 240$ times$=2400$ times

It will be seen that the multiplication by factors (2nd and 3rd modes) requires the fewest figures; it is therefore the more elegant, and, in addition to this, enables us, by altering the order of the factors, to apply an almost absolute verification.

8. Find the cost of 42 articles if 7 cost £7. 8s. 10d. $42 = 6 \times 7$, ∴ 42 articles cost 6 times as much as 7 cost = 6 × £7. 8s. 10d.

£7	8	10
		6
44	13	0

If I walk 76 miles in 4 days, how far shall I walk in 20 days? $20 = 5 \times 4$, ∴ in 20 days I shall walk 5 times as far as in 4 days, that is 5×76 miles $= 380$ miles.

9. Simplify $8 \times (16 + 5 - 3)$. $8 \times (16 + 5 - 3)$ means 8 times the quantity within the brackets, therefore we must first simplify $16 + 5 - 3$, which becomes $21 - 3$, or 18. Then 8×18, or 144, is the answer.

10. In multiplying two abstract numbers, it is better to choose as the multiplier that number which has the fewer significant figures.

Exercise XVIII.

(1) £47. 19*s.* 8¾*d.* × 371
(2) £612. 9*s.* 7¼*d.* × 802
(3) £8. 3*s.* 5½*d.* × 5049
(4) £9. 0*s.* 11*d.* × 62
(5) £50. 17*s.* 10½*d.* × 10010
(6) £471. 13*s.* 9¾*d.* × 7021

In two ways.
(7) £25. 7*s.* 4¼*d.* × 280
(8) £83. 15*s.* 3*d.* × 6400
(9) £90. 11*s.* 2½*d.* × 180
(10) £308. 19*s.* 1¼*d.* × 360
(11) £621. 14*s.* 9¾*d.* × 42
(12) £495. 6*s.* 6*d.* × 840

(13) 36724 × 37
(14) 5809 × 1209
(15) 8067 × 2109
(16) 53244 × 5005
(17) 178 × 16359
(18) 50070 × 829
(19) 69532 × 10101
(20) 69532 × 11001
(21) 69532 × 1010
(22) 69532 × 11111
(23) 69532 × 100100
(24) 69532 × 11011
(25) 26418 × 210
(26) 57390 × 4500
(27) 256080 × 120
(28) 7439 × 320
(29) 6428000 × 7200
(30) 4286 × 5500

(31) Find the daily wages of 367 men at 3*s.* 4½*d.* per day each.

(32) How much would this amount to in a year, leaving out 52 Sundays?

(33) Find the cost of 815 articles at 7*s.* 10¾*d.* each.

(34) Find the value of 3000 Venetian ducats at 9*s.* 5*d.* each.

(35) Find the value of 94375 Prussian thalers at 2*s.* 10¾*d.* each.

(36) Find the value of 1722 railway tickets at 1*s.* 7*d.* each.

(37) Find the value of a cwt. (112 lbs.) of sugar at 4¾*d.* per lb.

(38) Find the cost of a ton (20 cwt.) of iron nails at ¾*d.* per lb.

(39) What is the charge for translating 74 folios at 2*s.* 4½*d.* each?

(40) Find the cost of a lb. Troy (5760 grains) at 1¾*d.* per grain.

(41) What is the yearly rent of a cottage at 3*s.* 3*d.* a-week?

(42) Find the yearly rent of a terrace of 165 houses, of which each pays £136. 10*s.*

(43) Find the strength of an army consisting of 113 regiments of 947 men each.

(44) How many words are there on a page of 29 lines, each line containing 14 words?

(45) How many bricks are there in 306 yards of wall, each requiring 288 bricks?

(46) At the latitude of London, 1 degree (1^0) of longitude is very nearly 37 geographical miles. Find the length of the whole parallel (360^0).

(47) How many eggs are there in 3080 boxes, each containing 4769 eggs?

(48) How many grains in 68340 lbs. Avoirdupois (7000 grains each)?

(49) How many hours in a year of 365 days?

(50) How many inches will a wheel have travelled over in making 5317 turns, if the circumference of the wheel is 127 inches?

(51) How many yards in the equator, which is 24899 miles in length (1 mile = 1760 yards)?

(52) How many feet in the same (1 yard = 3 feet)?

(53) How many inches in the same (1 foot = 12 inches)?

(54) What is the issue of a newspaper in 13 weeks, the daily issue of which is 97428 copies?

(55) The Penny Cyclopædia consists of 27 vols., each vol. has, on an average, 512 pages, each page has two columns, and each column has 81 lines of about 8 words each. How many words are there in all?

(56) How many ounces in 1 ton Avoirdupois? (see Table.)

(57) How many inches in a mile?

(58) How many farthings in £1?

(59) How many pounds in 143 iron plates, each weighing 5 tons?

(60) Sound travels through air at the rate of 1130 feet per second. How many feet distant is a thunder-cloud if the report is heard 17 seconds after the flash is seen?

(61) Shew that $27043 \times 233 + 27043 \times 419 + 27043 \times 326 = 27043 \times 326 \times 3$.

(62) Shew that £7. 11*s*. 10*d*. × 233 + £7. 11*s*. 10*d*. × 419 + £7. 11*s*. 10*d*. × 326 = £7. 11*s*. 10*d*. × 326 × 3.

(63) Find the total cost of 43 articles at 5*s*. 11¼*d*. each, 107 articles at 13*s*. 8½*d*. each, 2160 articles at 3*s*. 9¾*d*. each, and 11 articles at £2. 4*s*. 7*d*. each.

(64) Find the total cost of 243 pieces of T cloths of 30 yards each, at 3¾*d*. per yard; 67 pieces of sheeting of 38 yards each, at 1*s*. 10¼*d*. per yard, 375 pieces of shirtings at 6*s*. 2*d*. per piece, and 18750 yards of fents at 1¼*d*. per yard.

(65) How much change shall I receive out of £100 after paying the following bill: 5 cwt. of sugar at 3¾*d*. per lb., a chest of tea containing 53 lbs. at 2*s*. 10½*d*. per lb., half a ton of Carolina rice at 3¼*d*. per lb., 2086 lbs. of raw Ceylon coffee at 5½*d*. per lb?

(66) A farmer sold 437 sheep at £1. 17*s*. 9*d*. each, and bought 23 bullocks at £12. 12*s*. each, and 19 calves at £7. 17*s*. 6*d*. each. How much has he left?

(67) If I spend 1*s*. 9*d*. a-day for a return ticket, travelling six days in the week, how much shall I save in a year (52 weeks) by taking an annual ticket for £25?

(68) I bought 843 articles at £1. 3*s*. 4½*d*. each, and sold them for £1000. What was my profit?

(69) I bought 756 articles for £195, and sold them at 5*s*. 7½*d*. each. What was my gain?

(70) I bought 400 dozen of wine at £1. 15*s*. 6*d*. per dozen, and retailed them at 3*s*. 6*d*. per bottle. Find my profit per dozen; also, in two ways, the profit on the whole.

(71) I bought 325 dozen of wine at 3*s*. 2*d*. per bottle. For how much must the whole be sold to gain £22. 10*s*.?

(72) A trader took out to China 24,178 pieces of grey shirting, of 29 yards each, at 2¼*d*. per yard; 12,089 pieces of print, of 58 yards each, at 4½*d*. per yard; 100,000 yards of flannel at 11½*d*. per yard, and 2400 dozen handkerchiefs at 1*s*. 7½*d*. per dozen. He brought back 3000 chests of tea, of 35 lbs. each, at 10*d*. per lb.; 16,498 lbs. of raw silk at £1. 3*s*. 10½*d*. per lb., and the remainder in cash. How much money did he bring back?

(73) If I pay £13. 13s. per quarter for rent, £2. 8s. 6d. per quarter for rates, £2. 18s. 9d. a-week for food, 11s. 3d. a-week for washing, £42 every half-year for school bills, £15. 12s. 6d. every half-year for life insurance, £35 a-year for clothes, £3. 10s. for coals for the summer half and £7. 7s. for the winter half-year, and wish to lay by £120 out of an income of £600 a-year, how much shall I have for other expenses?

(74) A publisher sells a certain book at 3s. 2d. per copy nett; of this, he pays to the printer, $9\frac{1}{4}d$. per copy; to the binder, $6\frac{3}{4}d$. per copy; to the author, a royalty of 9d. for every copy he sells. Of an edition of 1000 copies, he sells 853; the remainder are left on hand. Will he have lost or gained, and how much?

(75) I bought 2 tons of sugar at $3\frac{1}{4}d$. per lb.; 7 cwt. 21 lbs. got damaged. I sold the good sugar at 4d. a pound, and the damaged sugar at $2\frac{1}{2}d$. per pound. What was my gain or loss?

(76) A person mixed 23 gallons of Jamaica rum at 9s. 7d. per gallon, with 18 gallons of British rum at 7s. 5d. per gallon, and 30 gallons of water; he sold the mixture at 11s. 6d. per gallon. What was his total gain?

(77) A person bought 427 yards of cloth at 3s. 8d. per yard; for how much must he sell the whole so as to gain $7\frac{1}{2}d$. per yard?

(78) If 8 articles cost £4. 7s. $10\frac{1}{4}d$., what will 56 articles cost?

(79) If 9 men can dig 43 yards of trench in a given time, how much will 45 men dig in the same time?

(80) How much in double the time?

(81) If 3 men build a given wall in 14 days, how long would 1 man take to build it?

(82) If 20 men build a given wall in 8 days, how long will 10 men take to do it?

(83) And how long would 4 men take?

(84) If 3 articles cost 11s. $8\frac{1}{2}d$., what will 27 articles cost?

(85) If 8 chairs cost £2. 2s. 6d., what will 64 chairs cost?

(86) If one dozen pens cost $3\frac{1}{4}d$., what will the gross cost?

(87) A man bought 28 Bandana handkerchiefs at 7 for £1. 12s. $7\frac{1}{2}d$.; he sold them at 5s. 9d. each. Find his total profit.

(88) By mistake he entered this profit in his books among the losses; at the end of the month his books shewed a profit of only £83. 10*s*. What ought they to have shewn?

(89) In a given time 9 men can put up 67 yards of fence. How many yards can 63 men put up in treble the time?

(90) If 17 men can dig a given quantity of trench in a given time, how many men will be wanted to dig 4 times the quantity in a quarter of the time?

(91) If out of a sack of flour we make 43 shilling loaves, how many threepenny loaves could we have made of it?

(92) Simplify $17 \times (184 - 47 + 62)$.

(93) Simplify $302 \times (5 \times 17 - 3 \times 11)$; $597 \times (5 \times 17 - 9 \times 9)$.

(94) Simplify $6 \times 7 \times (11 \times 156 + 13 \times 84)$.

(95) Simplify $(81 + 317) \times (24 + 11)$.

(96) Simplify $6 \times 8 \times 9 \times 5 \times 4 \times 317$.

(97) Find the continued product of the first 9 numbers.

(98) Shew that $740625 \times 386 + 740625 \times 234 + 740625 \times 1671 + 740625 \times 176 + 740625 \times 809 + 740625 \times 546 = 740625 \times 546 \times 7$.

(99) Shew that £7. 8*s*. $1\frac{1}{2}$*d*. $\times$ 386 + £7. 8*s*. $1\frac{1}{2}$*d*. $\times$ 234 + £7. 8*s*. $1\frac{1}{2}$*d*. $\times$ 1671 + £7. 8*s*. $1\frac{1}{2}$*d*. $\times$ 176 + £7. 8*s*. $1\frac{1}{2}$*d*. $\times$ 809 + £7. 8*s*. $1\frac{1}{2}$*d*. $\times$ 546 = £7. 8*s*. $1\frac{1}{2}$*d*. $\times$ 546 $\times$ 7.

(100) A bankrupt owes £8745, and can pay 9*s*. $5\frac{3}{4}$*d*. in the £1. What are his assets?

(101) A railway company has 16,200 shares. Their expenses are £14,175. What must be the company's gross income to yield a dividend of 17*s*. 6*d*. per share?

(102) Multiply the sum of 438619 and 30405 by 198.

(103) Multiply the difference between 438619 and 30405 by 1098.

(104) Multiply the sum of 2815 and 365 by the difference between these two numbers.

(105) From 18 times the product of 519 and 98 take 7 times this product.

CHAPTER VII.

DIVISION.

1. How many *times* can I pay £4. 7*s.* 9½*d.* out of £26. 6*s.* 9*d.*?

£	s.	d.	
£26	6	9	
4	7	9½	1 time
21	18	11½	
4	7	9½	1 "
17	11	2	
4	7	9½	1 "
13	3	4½	
4	7	9½	1 "
8	15	7	
4	7	9½	1 "
4	7	9½	
4	7	9½	1 "
—	—	—	6 times exactly.

How many times are £5. 8*s.* 10*d.* contained in £43. 17*s.* 11*d.*?

£	s.	d.	
£43	17	11	
5	8	10	1 time
38	9	1	
5	8	10	1 "
33	0	3	
5	8	10	1 "
27	11	5	
5	8	10	1 "
22	2	7	
5	8	10	1 "
16	13	9	
5	8	10	1 "
11	4	11	
5	8	10	1
5	16	1	
5	8	10	1
0	7	3	8 times and 7*s.* 3*d.* over.

How many times are 112 lbs. contained in 800 lbs.?

800	
112	1 time
688	
112	1 "
576	
112	1 "
464	
112	1 "
352	
112	1 "
240	
112	1 "
128	
112	1 "
16	7 times and 16 lbs. over.

How many men will it require to dig 41 yards of trench in a given time if 1 man can dig 6 yards in that time?

41	
6	1 man
35	
6	1 "
29	
6	1 "
23	
6	1 "
17	
6	1 "
11	
6	1 "
5	6 men and 5 yards over.

Six men would in the given time leave 5 yards to be done, seven men would be able to do 1 yard more than required; we may choose either answer, but 7 men is the more correct. *Ans.* 7 men and 1 yard to spare.

The process by which we find the *number of times* that one quantity is contained in another is called Division, for which the symbol is ÷.

2. Learn by heart: *This sign* (÷) *is called* DIVIDED BY, *and has the following meaning: Find* HOW MANY TIMES *the quantity following the sign is contained in the quantity preceding the sign, and the answer will be* SO MANY TIMES.

3. Learn by heart: *In Division the quantity to be divided is called the* DIVIDEND, *that by which we divide is called the* DIVISOR, *and the answer is called the* QUOTIENT. *If anything is over it is called the* REMAINDER.

EXERCISE XIX.

(1) How many times are £5. 7*s.* 11*d.* contained in £16. 3*s.* 9*d.*?

(2) £13. 13*s.* 10*d.* ÷ £3. 8*s.* 5½*d.*

(3) If 1 article costs 12*s.* 7½*d.*, how many can I buy for £4. 8*s.* 4½*d.*?

(4) If I save £1. 4*s.* 10½*d.* a-week, how long shall I be in accumulating £9. 19*s.*?

(5) By what number must I multiply £8. 4*s.* 6*d.* to make £41. 2*s.* 6*d.*?

(6) To how many persons can I give £3. 15*s.* each out of £37. 10*s.*?

(7) If I can travel 65 miles a-day, how long will it take me to get over 520 miles?

(8) If one strip of carpet is 115 inches in length, how many strips can I cut from 1035 inches of carpet?

(9) How many regiments of 875 men each are there in 9625 men?

(10) How many yards of wall can I build with 448 bricks if each yard requires 64 bricks?

(11) How many times are £3. 8*s.* 10*d.* contained in £15?

(12) £37. 9*s.* 4*d.* ÷ £4. 13*s.* 7*d.*

(13) If 1 article costs £1. 5*s.* 9*d.*, how many can I buy for £8. 1*s.* 1*d.*?

(14) If I save £2. 12*s.* 6*d.* a month, how long will it take me to accumulate £25?

(15) Out of £18. 4*s.* 9*d.* I bought as many sheep as I could at £1. 15*s.* 6*d.* each, and with the remainder I bought a lamb. Find the cost of the lamb.

(16) Out of £13. 13*s.* I bought as many articles as I could at £1. 10*s.* each. How much more money shall I want to buy one more article?

(17) How many months of 28 days each are there in a year?

(18) If I start Oct. 4th and travel 85 miles a-day, on what day shall I reach a place 450 miles off?

(19) How many periods of 12 minutes are there in 1 hour?

(20) To how many persons can I give £68. 14*s.* 10*d.* if I have £600?

4. We see that Division is Cumulative Subtraction. Cumulative Subtraction can be worked by the help of Multiplication, which is Cumulative Addition.

How many times are £147. 10*s.* 3*d.* contained in £734. 8*s.*? We see that the answer depends chiefly upon the left-hand figure of the pounds—on the number of times, in fact, that 14 tens are contained in 73 tens. Now $5 \times 14 = 70$, hence we *guess* that the answer is 5 times. 5 times £147. 10*s.* 3*d.* = £737. 11*s.* 3*d.*

```
£147 10  3) £734  8  0 (5 times.
             737 11  3
            ----------
```

Our guess has in this case been too large, since we cannot take £737 from £734. Try 4 times:

```
£147 10  3) £734  8  0 (4 times.
             590  1  0
            ----------
             144  7  0
```

Answer. 4 times, and £144. 7*s.* over.

If the actual subtractions are made, it will be seen that we obtain the same result.

How many times are £36*s.* 8*s.* 4*d.* contained in £629. 11*s.* 7*d.*? We see at once that the divisor is contained more than 10 times in the dividend, because $10 \times £36 = £360$. Accordingly we may begin

by deducting 10 × £36. 8*s.* 4*d.* = £364. 3*s.* 4*d.*, which leaves a remainder of £265. 8*s.* 3*d.* Out of this we can take £36. 8*s.* 4*d.* 7 more times, because 7 × £36. 8*s.* 4*d.* = £254. 18*s.* 4*d.*, and this leaves £10. 9*s.* 11*d.* as *final* remainder.

£36 8 4)	£629	11	7	
	364	3	4	10 times.
	265	8	3	
	254	18	4	7 ,,
	10	9	11	17 times.

Answer. 17 times, and £10. 9*s.* 11*d.* over.

EXERCISE XX. (*a*).

(1) How many articles at £3. 7*s.* 10½*d.* each can be bought for £16. 19*s.* 4½*d.*?

(2) How many shares at £92. 12*s.* 6*d.* can be bought for £926. 5*s.*?

(3) The passengers in a railway carriage from London to Manchester paid together £6. 3*s.* 8*d.*; each ticket cost 15*s.* 5½*d.* How many passengers were in the carriage?

(4) £1103. ÷ £137. 17*s.* 6*d.*?

(5) A man's weekly expenses were: lodgings, 16*s.*; food, £1. 1*s.* 9*d.*; railway travelling, 5*s.*; washing, 2*s.*; sundries, 5*s.* 3*d.* How many weeks will £30 last him?

(6) One hundred and forty children are to be arranged in rows of 14 each. How many rows will there be?

(7) One hundred and forty-four children are to be seated in rows of 16 children each. How many rows will there be?

(8) A certain corps d'armée consisted of 10,835 men; each regiment had 859 private soldiers and 126 officers. How many regiments were there?

(9) How many such corps d'armée in an army of 184,195 men?

(10) A printer's office employs 9 compositors at £1. 16*s.* each, 3 pressmen at £1. 16*s.* each, 2 readers at £2. 10*s.* each, 1 overseer at £3, 2 boys at 11*s.* each a-week. How many weeks' wages can be paid with £276. 6*s.*?

(11) A sack of potatoes weighs 168 lbs. How many sacks in 1200 lbs.?

(12) A certain railway had to carry 9000 excursionists; each train could convey 456 passengers. How many trains must have been sent?

(13) A person earns £2. 7*s.* a-week, and spends £1. 12*s.* 6*d.* a-week. In how many weeks will he save £13. 1*s.*?

(14) How many years (of 365 days each) are there in 1000 days?

(15) £427. 19*s.* 5*d.* ÷ £63. 14*s.* $2\frac{1}{2}$*d.*

(16) £4000. ÷ £234. 5*s.* 6*d.*

(17) £876. 10*s.* ÷ £43. 16*s.* 6*d.*

(18) £1234. 5*s.* 6*d.* ÷ £89. 10*s.* 11*d.*

(19) £1000. ÷ £33. 6*s.* 8*d.*

(20) £500. ÷ £16. 13*s.* 4*d.*

5. If 1 article costs 4*s.* 7*d.*, how many can I buy for £28,450? On trial it would be found that the process employed above would be so long as to be practically useless. Some artifice for contracting the operation is therefore indispensable. (See Ch. IV. § 7.) Make a table as follows:

1 article costs	....	£0	4	7
10 articles cost	....	2	5	10
100 ,,	....	22	18	4
1000 ,,	....	229	3	4
10000 ,,	....	2291	13	4
100000 ,,	....	22916	13	4

By means of this table we see that we can buy 100,000 articles, and shall yet have some money remaining.

	£28450	0	0
	22916	13	4
1st remainder	5533	6	8

The table further shews us that with this remainder we can pay not only for 10,000, but for twice 10,000, or 20,000 articles. 20,000 articles cost 2 × £2291. 13*s.* 4*d.* = £4583. 6*s.* 8*d.*

	£5533	6	8	= 1st remainder
	4583	6	8	= cost of 20,000 articles
2nd remainder	950	0	0	

With this second remainder we cannot buy another 10,000, but we can buy much more than 1000 articles, and the question arises:

How many thousands? 1000 articles cost, by the table, £229. 3s. 4d. To make a rough guess as to the number of times this is contained in the second remainder, £950, we should examine the figures of highest value in each, which are respectively 2 hundreds and 9 hundreds; this leads us to suppose that we can pay for 4 thousands. On trial we find that 4 × £229. 3s. 4d. = £916. 13s. 4d.

	£950	0	0	= 2nd remainder
	916	13	4	= cost of 4000 articles
3rd remainder	33	6	8	

With this third remainder we can pay for 100 articles.

	£33	6	8	= 3rd remainder
	22	18	4	= cost of 100 articles
4th remainder	10	8	4	

With this fourth remainder we can buy 4 × 10 articles. 4 × £2. 5s. 10d. = £9. 3s. 4d.

	£10	8	4	= 4th remainder
	9	3	4	= cost of 40 articles
5th remainder	1	5	0	

With this fifth remainder we can buy 5 articles. 5 × 4s. 7d. = £1. 2s. 11d.

	£1	5	0	= 5th remainder
	1	2	11	= cost of 5 articles
6th remainder	0	2	1	

Consequently we can pay for 124,145 articles, and shall have 2s. 1d. over, which will not pay for another article.

Mod. op.:

4s.7d.)	£28450	0	0	
	22916	13	4	100000 articles
	5533	6	8	
	4583	6	8	20000 "
	950	0	0	
	916	13	4	4000 "
	33	6	8	
	22	18	4	100 "
	10	8	4	
	9	3	4	40 "
	1	5	0	
	1	2	11	5 "
Remainder......	0	2	1	124145 articles

Table.

	4	7	I.
2	5	10	X.
22	18	4	C.
229	3	4	M.
2291	13	4	XM.
22916	13	4	CM.

Answer. 124145 articles, and 2s. 1d. over.

EXERCISE XX. (*b*).

(1) How many times are £5. 18*s.* 7*d.* contained in £800. 8*s.* 9*d.*?

(2) How many articles can I buy for £2118. 18*s.* 3½*d.* if each costs £9. 15*s.* 3½*d.*?

(3) By what number must I multiply £15. 13*s.* 7*d.* to obtain £4280. 8*s.* 3*d.* for product?

(4) £4973. 11*s.* 2*d.* ÷ £30. 17*s.* 10*d.*

(5) If I put by £91. 7*s.* a-year, how long shall I take to accumulate £1370. 5*s.*?

(6) I invest £9287. 10*s.* in railway shares, each costing £92. 17*s.* 6*d.* and yielding a yearly income of £3. 10*s.* each. Find my total yearly revenue.

(7) How many Napoleons at 15*s.* 9*d.* each can I get for 189 Prussian thalers at 2*s.* 10*d.* each?

(8) (227 × £16. 11*s.* 4*d.*) ÷ (28 × 18*s.* 11*d.*).

(9) How many guineas are there in £510. 6*s.*?

(10) A earns £9. 2*s.* 6*d.* a-week, and spends £7. 5*s.* a-week. How long will he be in saving £150?

(11) A man's wages are £1. 17*s.* 6*d.* a-week; his wife earns 18*s.* a-week; his 2 sons earn 6*s.* 9*d.* a-week each. How long must the wages of the family remain unpaid to amount to £79. 7*s.*?

(12) How many sovereigns, half-sovereigns, crowns, half-crowns, florins, shillings, sixpences, fourpenny-pieces, threepenny-pieces, pennies, halfpennies, and farthings, an equal number of each, can be got from £358. 17*s.* 5*d.*?

(13) How many times are £3. 18*s.* 10*d.* contained in £429. 11*s.* 3*d.*?

(14) £5327. 3*s.* 5*d.* ÷ £6. 13*s.* 2*d.*

(15) A man's income is £2. 7*s.* 6*d.* a-week, and his expenditure, on an average, £3. 1*s.* 10*d.*, but he has £50 to begin with. How much a-week does he spend more than he gets? How many weeks will the £50 keep him out of debt? And how much will he be in debt after 100 weeks from the commencement?

(16) A certain book cost 7¾*d.* per copy for the paper, 4½*d.* for the printing, 5½*d.* for the binding. The total issue cost £110. 18*s.* 9*d.*

Of how many copies did it consist? And what was the profit on the whole issue if each copy was sold for 2*s.*?

(17) An omnibus costs to work, 5*s.* 6*d.* a-day for the driver, 5*s.* a-day for the conductor, 8*s.* 6*d.* a-week for the keep of each of 8 horses, 6*s.* 9*d.* a-week for sundries. These omnibuses run on Sundays. If the weekly expenses amount to £333. 11*s.* 3*d.*, how many omnibuses are there at work?

(18) How many times can we subtract £1. 3*s.* 7½*d.* from £78,492, and what will be over?

(19) If I have £18. 11*s.* 9*d.*, and buy as many books as I can at 5*s.* 3*d.* each, and with the remainder buy a slate, what did it cost?

(20) With an inheritance of £12,700, I bought as many shares at £92. 7*s.* 6*d.* as I could get. How many shares at £1. 9*s.* 9*d.* can I buy with the remainder?

(21) How many times must £3. 5*s.* 8½*d.* be added to £562. 12*s.* 11*d.* to make £14,000?

(22) Find my income if my income-tax at 7*d.* in £1 amounts to £14. 2*s.* 11*d.*?

(23) How many Napoleons at 14*s.* 11½*d.* each are equal to 718 Prussian thalers at 2*s.* 10¾*d.* each?

(24) How many pounds of tea at 2*s.* 8*d.* per lb. must be given in exchange for 112 lbs. of coffee at 1*s.* 2*d.* per lb., and 88 lbs. of raw sugar at 4*d.* per lb.?

(25) £1728. 18*s.* ÷ £4. 4*s.* 9*d.*
(26) £588. 7*s.* ÷ £4. 15*s.* 8*d.*
(27) £181. 8*s.* 7½*d.* ÷ 3*s.* 7½*d.*
(28) £330 ÷ 7*s.* 4*d.*
(29) £4559. 19*s.* 6*d.* ÷ £6. 4*s.* 3*d.*
(30) £4080 ÷ £5. 13*s.* 4*d.*
(31) £93. 15*s.* ÷ 2¼*d.*
(32) £16. 10*s.* 9*d.* ÷ 2*s.* 7½*d.*
(33) £5611. 1*s.* ÷ £5. 12*s.* 4*d.*
(34) £17914. 11*s.* 11¾*d.* ÷ £7. 8*s.* 10¼*d.*
(35) £734. 8*s.* 2*d.* ÷ £7. 1*s.* 1¼*d.*
(36) £1680 ÷ £5. 7*s.* 8*d.*
(37) £154. 10*s.* ÷ 4*s.* 3¾*d.*
(38) £3157. 15*s.* 1*d.* ÷ £7. 17*s.* 7¼*d.*
(39) £601000 ÷ £823. 1*s.* 4*d.*
(40) £237. 15*s.* 6*d.* ÷ 2¾*d.*
(41) £7425. 18*s.* 10½*d.* ÷ £24. 3*s.* 8½*d.*
(42) £90197. 14*s.* 10½*d.* ÷ £81. 3*s.* 7¾*d.*
(43) £8878. 3*s.* 9½*d.* ÷ £16. 3*s.* 5¾*d.*
(44) £999900 ÷ £99. 19*s.* 11¾*d.*

CHAPTER VIII.

DIVISION *(continued)*.

1. Distribute £9. 12*s.* 6*d.* equally among 3 persons. How much will each have?

The £9 will give £3 each, the 12*s.* will give 4*s.* each, and the 6*d.* will give 2*d.* each; therefore each person will have £3. 4*s.* 2*d.*

This question is indicated thus: £9. 12*s.* 6*d.* ÷ 3.

The symbol (÷) has thus a perfectly *new* interpretation, viz., the distribution of a given quantity into so many *equal parts.*

Learn by heart: *This sign* (÷) *is called* DIVIDED BY, *and bears* TWO *interpretations; 1st.* HOW MANY TIMES *is the* QUANTITY *following the sign contained in the quantity preceding the sign, and the answer will be* SO MANY TIMES; *2nd. Distribute the quantity before the sign into as many equal parts as is indicated by the* NUMBER *after the sign, and the answer will be* SO MUCH TO EACH PART.

2. Divide £357. 3*s.* 8*d.* between 2 persons. Dividing first the 3 hundred-pound notes, each person would receive 1 hundred-pound note, and there would be 1 hundred-pound note over; this we convert into 10 ten-pound notes, which, with the 5 ten-pound notes we have already, make 15 ten-pound notes, and these, divided among the 2 persons, give 7 ten-pound notes to each, and 1 ten-pound note over; converting this into 10 pounds, and adding the 7 pounds we have already, we have 17 pounds, of which we can give 8 pounds to each of the 2 persons, leaving 1 pound over; convert this into 2 half-sovereigns, and give 1 half-sovereign to each; now divide the 3 shillings, giving 1 shilling to each, and convert the 1 shilling over into 12 pence, which, with the 8 pence, give 20 pence, of which each person will have 10 pence. Total to each person, £178. 11*s.* 10*d.*

	£.	*s.*	*d.*
2)	357	3	8
	£178	11	10

Wording: 2 in 3, 1′, carry 1; in 15, 7′, carry 1; in 17, 8′, carry 1 (pound) = 2 half-sovs.; in 2, 1′; in 3, 1′, carry 1 (shilling) = 12 pence; in 20, 10′. *Ans.* £178. 11*s.* 10*d.*

Divide £743. 17*s.* 8¼*d.* among 3 persons.

	£.	*s.*	*d.*
3)	743	17	8¼
	£247	19	2¾

Wording: 3 in 7, 2′, carry 1; in 14, 4′, carry 2; in 23, 7′, carry 2; in 5, 1′, carry 2; in 27, 9′; in 8, 2′, carry 2; in 9, 3′. *Ans.* £247. 19*s.* 2¾*d.*

Divide £27,915. 7*s.* 8*d.* into 4 equal parts.

	£.	*s.*	*d.*
4)	27915	7	8
	£6978	16	11

Wording: 4 in 27, 6′, carry 3; in 39, 9′, carry 3; in 31, 7′, carry 3; in 35, 8′, carry 3; in 6, 1′, carry 2; in 27, 6′, carry 3; in 44, 11′. *Ans.* £6978. 16*s.* 11*d.*

Distribute £19,007. 6*s.* 7*d.* among 7 persons.

	£.	*s.*	*d.*
7)	19007	6	7
	£2715	6	7¾, ¾ over.

Wording: 7 in 19, 2′, carry 5; in 50, 7′, carry 1; in 10, 1′, carry 3; in 37, 5′, carry 2; in 46, 6′, carry 4; in 55, 7′, carry 6; in 24, 3′, and 3 farthings *over*. As we have no coin smaller than farthings, these 3 farthings must remain undistributed.

Distribute £252,047. 9*s.* 4¼*d.* among 6 persons.

	£.	*s.*	*d.*
6)	252047	9	4¼
	£42007	18	2½, $\frac{5}{4}$ over.

Wording: 6 in 25, 4′, carry 1; in 12, 2′; in 0, 0′; in 4, 0′, carry 4; in 47, 7′, carry 5; in 10, 1′, carry 4; in 49, 8′, carry 1; in 16, 2′, carry 4; in 17, 2′, and 5 farthings over.

Exercise XXI.

(1) Distribute £17,419. 8*s.* 9*d.* between 2 persons.

(2) What is the half of £73 13*s.* 9*d.*?

(3) Divide £68,422. 5*s.* 9¾*d.* into 3 equal parts.

(4) What is the third part of £11,019. 5*s.* 6*d.*?

(5) What sum of money can be subtracted 4 times exactly from £3509. 7*s.* 6*d.*?

(6) What is the quarter of £1007. 1*s.* 2*d.*?

(7) If 5 shares cost £7804. 6*s.* 8*d.*, what is the cost of each?

(8) What is the fifth part of £19,001. 2*s.* 7¼*d.*?

(9) If I spend £1111 in six years, how much is that for 1 year?

(10) What is the sixth part of £14. 9*s.* 3*d.*?

(11) What do I spend each day if I spend £19. 3*s.* 3*d.* in a week of 7 days?

(12) What is the seventh part of £85. 14*s.* 8½*d.*?

(13) If the wages of 8 men amount to £5. 9*s.* 8*d.*, what will each receive?

(14) What is the eighth part of £19. 7*s.* 6*d.*?

(15) If 9 articles cost £8. 7*s.* 5¼*d.*, what does each cost?

(16) What is the ninth part of £318. 14*s.* 0¾*d.*

(17) Distribute £519. 6*s.* 7*d.* equally among 3 persons.

(18) Divide £22,865. 9*s.* 11*d.* among 7 persons.

(19) What is the largest sum of money that can be subtracted 9 times *from £1000*?

(20) What number subtracted 8 times from £417. 6*s.* 5½*d.* will leave a remainder of £19. 11*s.* 7¼*d.*?

(21) £43750. 8*s.* 4*d.* ÷ 2	(26) £1050. 10*s.* 6*d.* ÷ 7
(22) £1316. 9*s.* 7*d.* ÷ 3	(27) £1050. 10*s.* 6*d.* ÷ 8
(23) £5000 ÷ 4	(28) £1050. 10*s.* 6*d.* ÷ 9
(24) £1706. 8*s.* 6*d.* ÷ 5	(29) £473. 7*s.* 5*d.* ÷ 7
(25) £1050. 10*s.* 6*d.* ÷ 6	(30) £751. 1*s.* 1*d.* ÷ 8

3. Distribute 718 eggs into 2 baskets.

```
    C. X. I.
2)  7  1  8
    3  5  9  eggs.
```

Distributing the 7 hundreds, we put 3 hundreds into each basket, and have 1 hundred over; 1 hundred is 10 tens, and 1 ten to it makes 11 tens, which yields 5 tens for each basket, and 1 ten over, which, with the 8 units, makes 18 units, of which 9 go to each basket. *Ans.* 359 eggs to each basket.

Wording: 2 in 7, 3′, carry 1; in 11, 5′, carry 1; in 18, 9′.

Arrange 329 marbles into 7 equal heaps.

```
7) 329
   47
```

Ans. 47 marbles in each heap.

Arrange 329 marbles into 8 equal heaps.

```
8) 329
   41 and 1 over.
```

Ans. 41 marbles in each heap and 1 marble over.

Arrange 329 marbles into 9 equal heaps.

```
9) 329
   36 and 5 over.
```

Ans. 36 marbles in each heap and 5 marbles over.

4. Test of accuracy by casting out nines.


```
8) 435265
   54408 and 1 over.
```

Cast out nines from the dividend and place the result (7) in the upper space of the cross in the margin. Cast out nines from the divisor and the quotient, placing the results in the right and left hand spaces. Multiply these results (8 × 3 = 24), and add in the

remainder (1); (24 + 1 = 25, 7), and write this in the lower space. If the figures in the upper and lower spaces do not agree, there must be an error in the work.

EXERCISE XXII.

(1) Divide 17412 things into 2 equal parts.

(2) What is the half of 358 things?

(3) Divide 71235 things into 3 equal parts.

(4) What is the third part of 72861?

(5) What number of things can be subtracted four times exactly from 711136 things?

(6) What is the quarter of 1097324?

(7) If 5 equal baskets contain together 3125 apples, what will 1 basket contain?

(8) What is the fifth part of 3116845?

(9) Divide a journey of 528 miles into 6 equal stages.

(10) What is the sixth part of 2034?

(11) A certain railway guard travels 2303 miles a-week. How much is that a-day?

(12) What is the seventh part of 5565?

(13) If 8 horses can draw a load of 3416 lbs., what can 1 horse draw?

(14) What is the eighth part of 137904?

(15) If 9 regiments contain 8433 men, how many are there in each?

(16) What is the ninth part of 1000008?

(17) Distribute 92763 cartridges among 11 regiments.

(18) What is the eleventh part of 135795?

(19) If 12 volumes have 6084 pages, how many are there in each?

(20) Find the twelfth part of 122436.

(21) Divide 7034519 separately by 2, 3, 4, 5, 6, 7, 8, 9, 11, 12.

(22) Divide each of the following numbers by 12:

a. 1000000	*d.* 1111111	*g.* 106000
b. 4197641	*e.* 167625	*h.* 17000000
c. 42050000	*f.* 4065000	*k.* 9052500

(23) What sum of money must be multiplied by 2 to yield £317. 19*s.* 6*d.*?

(24) What number must be multiplied by 2 to give 173598?

(25) What sum of money must be taken 3 times to yield £1000?

(26) What number must be repeated 3 times to give 14367?

(27) What sum of money multiplied by 4 will amount to £1719. 3s. 6d.?

(28) What number multiplied by 4 will give 17003068?

(29) What was the value of each collection, if 5 collections yielded £317. 16s. 10½d.?

(30) How long is each side of a regular pentagon, if the whole perimeter is 935 inches?

(31) What sum of money must be multiplied by 6 to yield £4379?

(32) What number must be taken 6 times to give 14382?

(33) What is that sum of money which multiplied by 7 gives £962. 5s. 0½d.

(34) Find the number which multiplied by 7 yields 100569.

(35) What sum of money repeated 8 times will yield £19,000?

(36) What number multiplied by 8 will give 5371016?

(37) What sum of money multiplied by 9 will amount to £384. 15s. 11¼d.

(38) What number multiplied by 9 will give 123456789?

(39) What sum of money multiplied by 11 will give £38,020. 4s. 9½d.?

(40) What number taken 12 times will give 207000?

(41) If 56 articles cost £30. 14s. 11¾d., what will 8 articles cost?

(42) If 45 men dig 430 yards in a given time, how much will 9 men dig in the same time?

(43) If a gross cost 6s. 9d., what will a dozen cost?

(44) If 10 men take 16 days to build a wall, how long will 40 men take?

(45) A man sold 42 Bandana handkerchiefs for £8. 2s. 9d., making a profit of £1. 1s. What did each piece of 7 handkerchiefs cost him?

5. *Q.* What is the eighth part of 24 apples? *A.* 3 apples.

Q. Why? *A.* Because there are 8 threes in 24.

Q. How many times are 8 apples contained in 24 apples? *A.* 3 times.

Q. Why? *A.* Because there are 3 eights in 24.

Teacher. If, therefore, 24 is divided by 8, the quotient will be 3, whichever of the two interpretations (§ 1) of the sign (÷) we adopt; but the name to be attached to the quotient will depend on the interpretation required by the nature of the question; thus here the answer is in one case 3 *apples*, and in the other 3 *times*. In future, then, we may say, $24 \div 8 = 3$, without necessarily choosing between the two interpretations.

6. Distribute £718,436. 3*s.* 9¼*d.* amongst 317 persons.

If we apply the reasoning of § 2, we meet at once with the practical difficulty that we only know the multiplication table up to twelves; we therefore make a multiplication table for this occasion, thus:

$$\begin{array}{r}
1 \times 317 = 317 \\
2 \times 317 = 634 \\
3 \times 317 = 951 \\
4 \times 317 = 1268 \\
5 \times 317 = 1585 \\
6 \times 317 = 1902 \\
7 \times 317 = 2219 \\
8 \times 317 = 2536 \\
9 \times 317 = \underline{2853} \\
14265
\end{array}$$

N.B. The accuracy of the table may be tested by adding the nine products, and also multiplying the last product (the 9 times) by 5; these two results should agree.

Successive stages of the sum.

$$\begin{array}{llllll}
 & \text{CM.XM.M.C.X.I.} & \text{S.} & \text{D.F.} & \\
317) & 7\ 1\ 8\ 4\ 3\ 6 & 3 & 9\tfrac{1}{4} & (\\
\end{array}$$

$$\begin{array}{lllll}
 & & & & \text{M.} \\
317) & 718436 & 3 & 9\tfrac{1}{4} & (2 \\
 & \underline{634} & & & \\
 & \ \ 84 & & &
\end{array}$$

$$\begin{array}{lllll}
 & & & & \text{M.C.} \\
317) & 718436 & 3 & 9\tfrac{1}{4} & (2\ 2 \\
 & \underline{634} & & & \\
 & \ \ 844 & & & \\
 & \ \ \underline{634} & & & \\
 & \ \ 210 & & &
\end{array}$$

If 7 hundred-thousand-pound notes are to be distributed among 317 persons, there will not be a note a-piece; we therefore change them into notes of ten-thousand pounds, of which the 7 will yield 70, and 1 more which we have makes 71; these again will not yield 1 ten-thousand-pound note to each person. Converting these into thousand-pound notes, we obtain, with the 8, 718 thousand-pound notes; now we see by the table that twice 317 is less than 718, but that 3 times 317 is greater than 718, we therefore can give 2 thousand-pound

Successive stages of the sum.

```
                     M.C.X.
317) 718436   3   9¼ (226
     634
     844
     634
     2103
     1902
      201
                     M.C.X.I.
317) 718436   3   9¼(2266
     634
     844
     634
     2103
     1902
      2016
      1902
       114
         2
       228
                     M.C.X.I. £½.S. D.
317) 718436   3   9¼ (2266  —7  2¼
     634
     844
     634
     2103
     1902
      2016
      1902
       114
         2
      2283
      2219
        64
        12
       768
         9
       777
       634
       143
         4
       572
         1
       573
       317
       256
```

notes to each person, and shall have *some* thousand-pound notes over. Twice 317 = 634; subtracting this number from 718, leaves 84 thousand-pound notes, which, with the 4 hundred-pound notes, yield 844 hundred-pound notes, of which, by similar reasoning, we can give 2 hundred-pound notes to each, leaving us 210 hundred-pound notes over. Converting these into ten-pound notes, we obtain, with the 3, 2103 ten-pound notes. Consulting the table, we find we can give each person 6 ten-pound notes, and subtracting 6 × 317, or 1902, we have 201 ten-pound notes over. These, with the 6 pounds, make 2016 pounds, enabling us to give 6 pounds to each person, and leaving 114 pounds. To distribute these, we must change them into smaller coin, viz. half-sovereigns. Since each pound equals 2 half-sovs., multiply 114 by 2, yielding 228 half-sovs., which, being less than 317, must be again changed into shillings, and, with the 3, will yield 2283 shillings. We can now give each person 7 shillings, leaving 64 shillings. These changed into pence will yield 12 × 64 = 768 pence, and adding in the 9*d.*, 777 pence. Of these, we can give 2 pence to each person, leaving 143 pence. Multiplying these by 4 to convert them into farthings, and adding in the 1 farthing we have already, we get 573 farthings, of which we can give 1 farthing to each person, leaving 256 farthings. This remainder must be left undistributed. Hence the answer is, £2266. 7*s.* 2¼*d.* to each person, and 256 farthings over.

Table.

$1 \times 137 =$	137
$2 \times 137 =$	274
$3 \times 137 =$	411
$4 \times 137 =$	548
$5 \times 137 =$	685
$6 \times 137 =$	822
$7 \times 137 =$	959
$8 \times 137 =$	1096
$9 \times 137 =$	1233
	6165

Second meaning.

7. Distribute 48729 marbles among 137 people. How many *marbles* to each? 1st. Make a table as before. 2nd. Analyse the dividend: 48729 = 4 ten thousands and 8729 units over = 48 thousands and 729 units over = 487 hundreds and 29 units over = 4872 tens and 9 units over (p. 7). 3rd. Proceed as follows: If 4 ten thousands are to be distributed amongst 137 persons, there will not be 1 ten thousand a-piece; the 48 thousands also will not yield 1 thousand a-piece. If the 487 hundreds are distributed amongst the 137 persons, we can give (see table) 3 hundreds a-piece, but not 4; subtracting 3 times 137 = 411 (see table) from 487 leaves 76 hundreds = 760 tens, which, with the 2 tens we have already, make 762 tens; of these, we can give (see table) 5 tens to each, leaving 77 tens = 770 units, which with the 9 units we have already, make 779 units; of these we can give (see table) 5 units to each,

Successive stages of the sum.

```
           C.
137) 48729 (3
     411
      76
```

```
           C.X.
137) 48729 (35
     411
      762
      685
       77
```

First meaning.

How many *times* are 137 marbles contained in 48729 marbles? 1st. Make the same table as in the margin. 2nd. Proceed as follows: We wish to take out at one subtraction as many times 137 as we can. It is clear that we can take 10×137 or 1370; we can also take it 100 times or 13700, but not 1000 times, for $1000 \times 137 = 137000$, which is more than the dividend. Now the question arises, Can we take more than 1 hundred at one subtraction? From the table, we see that $200 \times 137 = 27400$, $300 \times 137 = 41100$, but $400 \times 137 = 54800$, which is larger than the dividend. Subtracting, therefore, 300×137, we have 7629 units over.

```
137) 48729 (300 times
     41100
      7629 over
```

Out of this remainder we again wish to take at one subtraction as many times 137 as we can. We cannot take it another hundred times, or else we could have taken it 400

leaving 94 units over. The answer, therefore, is 355 *marbles* to each person, and 94 marbles over.

times at the last subtraction. By reasoning as above, we see from the table that we can take it 50 times.

```
137) 48729
     41100 = 300 times
      7629
      6850 —  50 times
       779 over
```

From the table, we see again that we can take it out 5 times more.

```
137) 48729
     41100 = 300 times
      7629
      6850 =  50 times
       779
       685 =   5 times
Remainder, 94   355 times
```

Ans. 355 *times* and 94 marbles over.

Successive stages of the sum.

```
              C.X.I.
137) 48729 (3 5 5
     411
      762
      685
       779
       685
        94
```

Casting out nines:

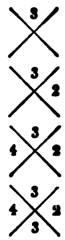
Dividend, 4, 12, 3, 10, 1, 3′ ...

Divisor, 1, 4, 11, 2′ ....

Quotient, 3, 8, 13, 4′ ....

Remainder, 4 ....

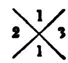
2 × 4 = 8; 8, 12, 3′ ....



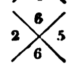
On comparing these two processes, we notice (1st) that the numerical values of the quotient and remainder are the same in each, the only difference being in the *meaning* of the quotient, as we might have expected from § 5; (2nd) that the form under the second meaning uses fewer figures. We shall adopt this method in all cases of division, except where the divisor is a compound number, in which case we must follow the method of Chapter VII. We must, however, take especial care to interpret the quotient rightly, and in this we can only be guided by the answer demanded in the question.

N.B. Supposing an error to be indicated by this test, it is not necessary to go through the whole work again, as each line can be tested separately.

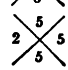
For the 1st remainder, cast out as though the dividend were 48700, the divisor 137, the quotient 300, and the remainder 7600 ......



For the 2nd remainder, cast out as though the dividend were 7620, the divisor 137, the quotient 50, and the remainder 770.........



For the 3rd remainder, cast out as though the dividend were 779, the divisor 137, the quotient 5, and the remainder 94............



8. Divide 1876706 by 7598.

Table.

$1 \times 7598 = 7598$
$2 \times 7598 = 15196$
$3 \times 7598 = 22794$
$4 \times 7598 = 30392$
$5 \times 7598 = 37990$
$6 \times 7598 = 45588$
$7 \times 7598 = 53186$
$8 \times 7598 = 60784$
$9 \times 7598 = 68382$
341910

G.X.I.

```
7598) 1876706 (2 4 7
      15196
      -----
       35710
       30392
       -----
        53186
        53186
        -----
        — —
```

```
7598) 1876706
      1519600 = 200 times
      -------
       357106
       303920 =  40 times
       ------
        53186
        53186 =   7 times
        -----     ---
        — —     247 times exactly
```

Therefore, if 1876706 things are distributed into 7598 equal lots, there will be 247 things in each lot. Again, if 1876706 things be distributed into lots of 7598 things each, there will be 247 lots, or, which is the same thing, $7598 \times 247 = 1876706$.

Verification:

```
   7598
    247
-------
1519600
 303920
  53186
-------
1876706
```

Comparing this multiplication with the division above, we see that the several addenda of the one are the successive subtrahends of the other, as might have been anticipated, seeing that Multiplication is shortened Addition, and Division is shortened Subtraction.

In such verification the remainder, if any, must be added to the product. Thus, to verify the example in § 7:

```
             137
             355
           -----
           41100
            6850
             685
           -----
           48635
Remainder,    94
           -----
           48729
```

Divide 14634167 by 3578.

$1 \times 3578 = 3578$	M.C.X.I.
$2 \times 3578 = 7156$	3578) 14634167 (4 0 9 0
$3 \times 3578 = 10734$	14312
$4 \times 3578 = 14312$	32216
$5 \times 3578 = 17890$	32202
$6 \times 3578 = 21468$	147
$7 \times 3578 = 25046$	
$8 \times 3578 = 28624$	*Ans.* 4090 times or things, and 147 things over.
$9 \times 3578 = 32202$	
161010	

N.B. After the first subtraction, we find in "bringing down" the 1 hundred that 3578 is not contained in 3221; we must therefore be very careful to register this in the quotient by putting a 0 in the hundreds' place. Similarly, after the second subtraction, we find that there are no units, and we again put a 0 in the units' place of the quotient. This remark shews the importance of not discarding too soon the headings, I. X. C., &c. When considerable facility has been acquired, these, as well as the marginal table, may be omitted. In the method given under the first meaning (p. 91), this difficulty would not have arisen.

Exercise XXIII.

(1) Distribute £17,802. 19*s.* 6¼*d.* among 371 persons.

(2) What sum of money must be multiplied by 135 to yield £2801. 10*s.* 7½*d.*?

(3) Distribute £643. 10*s.* 2½*d.* among 815 persons.

(4) What sum of money can be subtracted 750 times exactly from £706. 5*s.*?

(5) £6043. 11*s.* 7*d.* ÷ 892.

(6) If I spend £507. 19*s.* 2*d.* a-year, how much is that a-day?

(7) If I spend £1068. 5*s.* 6*d.* a-year, how much is that a-week?

(8) Divide £7003. 3*s.* 10¾*d.* into 1867 equal parts.

(9) If I invest £53,833. 5*s.* in 1234 shares, what is that per share?

(10) If I pay £323. 18*s.* 1*d.* for 764 pieces of calico of 37 yards per piece, how much does each piece cost, and how much a yard?

(11) 83094027 ÷ 9784. (12) 140940000 ÷ 13417.

(13) 140940000 ÷ 130417. (14) 8375808125 ÷ 18368.

(15) 22505600 ÷ 17312. (16) 3491938567017 ÷ 857928.

(17) How many bags of 819 marbles can I fill out of 20000 marbles?

(18) Divide the sum of 71 × 2117 and 17 × 1711 by 29.

(19) Divide the difference between the same quantities by 58.

(20) What sum of money subtracted 94 times from £848. 6*s.* 5*d.* will leave £62. 4*s.* 11*d.*?

(21) How much a-week may I spend out of an income of £374. 12*s.* 2*d.* a-year, to save 75 guineas a-year?

(22) £118,043. 19*s.* $7\frac{1}{4}$*d.* ÷ 269.

(23) £1000000 ÷ 89. (24) £1000000 ÷ 267.

(25) £1000000 ÷ 1869. (26) £1000000 ÷ 9345.

(27) £1000000 ÷ 84105. (28) £1000000 ÷ 925155.

(29) If 4204800 ounces of provisions are supplied to a regiment of 960 men, giving 12 ounces a-day to each man, how long will the supply last?

(30) How long would it last if each man had 60 ounces a-day?

(31) If the regiment were reduced to 800 men, each man having 24 ounces a-day, how long would it last?

(32) If at the age of 29 I begin business with a capital of £4500, and wish to retire at the age of 60 with a capital of £20,000, what yearly addition must I make to my capital?

(33) If a million bricks be required, and we have 77331 already, how many loads of 407 bricks each are wanted to complete the number?

(34) What will be the charge for translating 25344 words, at the rate of 1*s.* $7\frac{1}{2}$*d.* per folio of 72 words?

(35) 248073019 ÷ 43017. (36) 943867315 ÷ 12604.

(37) I sold 1512 articles for £690. 7*s.* 6*d.*, making a total profit of £123. 7*s.* 6*d.* Find the cost of each.

(38)* What sum of money must be multiplied by 623 to yield £733. 6*s.* $5\frac{1}{2}$*d.*?

(39)* What sum of money must be multiplied by 1246 to yield £733. 6*s.* $5\frac{1}{2}$*d.*?

(40)* What sum of money must be multiplied by 623 to yield £366. 13*s.* $2\frac{3}{4}$*d.*?

(41) 12357096 ÷ 419 (47) 20969 ÷ 13

(42) 214583206 ÷ 42576 (48) 100004 ÷ 17

(43) 366413796 ÷ 45796 (49) 1010100 ÷ 19

(44) 166944509 ÷ 23509 (50) 1000000 ÷ 37

(45) 19639 ÷ 23 (51) 3000000 ÷ 111

(46) 39278 ÷ 46 (52) 458629725 ÷ 9625

* Examine and compare the quotients in these three.

EXERCISE XXIV.

(1) Find the sum of 578, 364, 927, 9768.

(2) What number exceeds 578 by 344?

(3) From what sum of money must £42. 11*s*. 8*d*. be deducted to leave £57. 8*s*. 4*d*.?

(4) There are two numbers; the less is 7109, and their difference is 591. Find the greater.

(5) From what number must 316 be taken away to leave 518?

(6) If from a certain number 75 is taken, 89 is left. Find the number.

(7) Of two partners A and B, A contributes £520 less than B, whose share is £965. Find the total capital.

(8) 493 exceeds a certain number by 121. Find the number.

(9) What number falls short of 1096 by 421?

(10) What number is that to which 2768 must be added to give 10000.

(11) There are two numbers; the greater is 16520, and their difference is 3736. Find the less.

(12) What number increased by 2743 becomes 12000?

(13) Find the product of 69 and 237.

(14) What number contains 328 exactly 328 times?

(15) From what number can 5704 be taken exactly 104 times?

(16) From what number can 847 be taken 307 times, leaving a remainder of 49?

(17) Of what number is 53 the 7th part?

(18) What number divided by 97 gives 204?

(19) What is the 235th part of 141235?

(20) By what number must 397 be multiplied to give 170710?

(21) The product of two numbers is 4539, one factor is 51. Find the other.

(22) Given divisor 9373, quotient 103. Find the dividend.

(23) Given dividend 9373, quotient 103. Find divisor.

(24) Given dividend 9373, divisor 103. Find quotient.

(25) What number taken 103 times gives 965419?

(26) Given dividend 99201815, quotient 208, remainder 1207. Find divisor.

CHAPTER IX.

CONTRACTED OPERATIONS.

1. Multiply 5879 by 364.

```
                Full form.          Contracted form.
                   5879                  5879
                    364                   364
   \ 8 /         -------               ------
  2  X  4        1763700               17637
   / 8 \          352740                35274       a
                   23516                 23516
                 -------               -------
                 2139956               2139956

        or,        5879       or,        5879
                    364                   364
                  -----                 -----
                  23516                 23516
                 352740                35274        b
                1763700               17637
                -------               -------
                2139956               2139956
```

By carefully comparing the full with the contracted forms, it will be seen that the effect of the ciphers to the right of the several products can be produced by merely leaving blanks in their places. The form most commonly used is that marked *b*, but that marked *a* may also be practised with advantage.

Multiply 480730 by 5070.

```
                Full form.          Contracted form.
                  480730                480730
                    5070                  5070
   \ 3 /      ----------            ----------
  4  X  3     2403650000            2403650
   / 3 \        33651100              33651100
              ----------            ----------
              2437301100            2437301100

       or,        480730      or,       480730
                    5070                  5070
                --------              --------
                33651100              33651100
              2403650000            2403650
              ----------            ----------
              2437301100            2437301100
```

Should any difficulty be experienced, the places of the ciphers may at first be indicated by dots, thus:

387490 × 20400.

```
   387490
    20400
---------
154996000
774980....
----------
7904796000
```

This process should be still further contracted, thus:

```
  38749(0
    204(00
----------
 154996
77498
----------
7904796000
```

In this case we have multiplied only the significant figures, affixing to the ultimate product the final ciphers of both the factors, the reason for which follows directly from Ch. VI. § 5.

Exercise XXV.

(1) 8417 × 394
(2) 27349 × 5618
(3) 108912 × 4798
(4) 247863 × 365
(5) 68354 × 842
(6) 92731 × 516
(7) 430597 × 118
(8) 2413058 × 4237
(9) 48596 × 3420
(10) 68043 × 5070
(11) 4235700 × 8005
(12) 73050 × 9010
(13) 20014 × 1050
(14) 68000000 × 45000
(15) 2076980 × 30840
(16) 385604000 × 10500
(17) 30420 × 103684700
(18) 437598000 × 4601700
(19) 2804 × 43090
(20) 5498600 × 6420
(21) 6200 × 70800 × 9500
(22) 5860 × 2045 × 902 × 1000

2. Form (*a*) of § 1 readily admits of a still further contraction by "adding in" the figures of the last line of multiplication as they are obtained.

```
   5879
    364
-------
17637..  (a)
 35274.  (b)
-------
2139956
```

Wording in full: (After having obtained lines (*a*) and (*b*) in the usual way), 4 times 9 is **36′**, carry 3; 4 times 7 is 28, and 3 is 31, and 4 is **35′**, carry 3; 4 times 8 is 32, and 3 is 35, and 7 is 42, and 7 is **49′**, carry 4; 4 times 5 is 20, and 4 is 24, and 2 is 26, and 3 is **29′**, carry 2; 2 and 5 is 7, and 6 is **13′**, carry 1; 1 and 3 is 4, and 7 is **11′** carry 1; 1 and 1 is **2′**.

Wording to be used: **36′**, carry 3; 28, 31, **35′**, carry 3; 32, 35, 42, **49′**, carry 4; 20, 24, 26, **29′**, carry 2; 7, **13′**, carry 1; 4, **11′**; carry 1, **2′**.

Exercise XXVI.

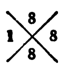
(1) 68497 × 5268
(2) 427906 × 6804
(3) 72840 × 5093
(4) 476927 × 20060
(5) 2748 × 16900
(6) 5080000 × 62490
(7) 287963 × 57846
(8) 6248 × 6248
(9) 5743 × 606
(10) 58743 × 871

(11) Find the continued product of 48212, 17 and 19.

3. Multiply 68437 by 17.

Full form.

```
  68437
     17
-------
 68437.
 479059
-------
1163429
```



Contracted form.

```
68437. × 17
-----------
1163429
```

Wording in full: 7 times 7 are 49, put down 9 and carry 4; 7 times 3 are 21, and 4 are 25, and 7 are 32, put down 2 and carry 3, &c.

Wording to be used: **49′**, carry 4; 21, 25, **32′**, carry 3; 28, 31, **34′**, carry 3; 56, 59, **63′**, carry 6; 42, 48, **56′**, carry 5; **11′**.

It will be found a safeguard against error to dot each figure as it is used.

Exercise XXVII.

(1) 438592 × 17
(2) 6017839 × 16
(3) 60857410 × 15
(4) 79845103 × 18
(5) 142587623 × 13
(6) 98043527 × 14
(7) 3260625 × 16
(8) 54370650 × 19
(9) 58472 × 13
(10) 8345620 × 17
(11) 287546 × 15
(12) 7863521 × 18
(13) 23781 × 14
(14) 4236541 × 17
(15) 1835429 × 13
(16) 987654 × 16
(17) 320070 × 1400
(18) 100936 × 17000
(19) 10608000 × 1300
(20) 234500 × 16000

4 Multiply 358934 by 103.

Full form.	Contracted form.
358934	358934.. × 103
103	36970202
358934..	
1076802	
36970202	

Wording: 12′, carry 1; 9, 10′, carry 1; 27, 28, 32′, carry 3; 24, 27, 30′ carry 3; 15, 18, 27′, carry 2; 9, 11, 19′, carry 1; 6′; 3′.

Multiply 4206008 by 10004.

Full form.	Contracted form.
4206008	4206008.... × 10004
10004	42076904032
4206008....	
16824032	
42076904032	

Wording: 32′, carry 3; 0, 3′; 0′; 24′, carry 2; 0, 2, 10′, carry 1; 8, 9 16′, carry 1; 7′; 0′; 2′; 4′.

Multiply 6749840 by 10600.

Full form.	Contracted form.
674984(0	6749840.. × 10600
106(00	71548304000
674984..	
4049904	
71548304000	

Wording : 24′, carry 2; 48, 50′, carry 5; 54, 59, 63′, carry 6; 24, 30, 38′ carry 3; 42, 45, 54′, carry 5; 36, 41, 45′, carry 4; 11′, carry 1; 7′. Now pu on the three ciphers.

EXERCISE XXVIII.

(1) 7563124 × 101
(2) 56342901 × 1001
(3) 290076358 × 10001
(4) 624345 × 102
(5) 82654352 × 103
(6) 9432654 × 107
(7) 20560078 × 1007
(8) 254836241 × 104
(9) 58364212 × 111
(10) 246314 × 1011
(11) 58432631 × 108
(12) 857320507 × 1005
(13) 25603054 × 1008
(14) 8946237 × 19
(15) 8946237 × 109
(16) 8946237 × 1009
(17) 27438956 × 106
(18) 537609000 × 1020
(19) 29330 × 170 × 1070 × 10070
(20) 358740100 × 10800

5. Multiply 437062 by 21.

Full form.

```
 437062
     21
 ------
 437062
874124.
-------
9178302
```

Contracted form.

$\dot{4}\dot{3}\dot{7}\dot{0}\dot{6}\dot{2} \times 21$

9178302

Wording in full: Once 2 is **2′**; twice $\dot{2}$ are 4, and 6 are **10′**, carry 1; twice $\dot{6}$ are 12, and 1 are 13, and $\dot{0}$ are **13′**, carry 1; twice $\dot{0}$ is 0, and 1 is 1, and $\dot{7}$ is **8′**; twice $\dot{7}$ are 14, and $\dot{3}$ are **17′**, carry 1; twice $\dot{3}$ are 6, and 1 are 7, and $\dot{4}$ are **11′**, carry 1; twice $\dot{4}$ are 8, and 1 are **9′**.

Wording to be used: **2′**; 4, **10′**, carry 1; 12, **13′**, carry 1; 0, 1, **8′**; 14, **17′**, carry 1; 6, 7, **11′**, carry 1; 8, **9′**.

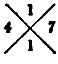
EXERCISE XXIX.

(1) 52019763 × 21
(2) 68320094 × 31
(3) 4107618 × 41
(4) 5090090 × 51
(5) 16487312 × 61
(6) 43506 × 71
(7) 728943 × 81
(8) 19312650 × 91
(9) 32863400 × 610
(10) 52900 × 4100
(11) 12468000 × 71000
(12) 48760200 × 3100
(13) 27341 × 71
(14) 27341 × 17
(15) 376859 × 31
(16) 376859 × 13
(17) 419068 × 15
(18) 419068 × 51
(19) 206867 × 41
(20) 206867 × 14

6. Multiply 7849624 by 6001.

Full form.

```
     7849624
        6001
     -------
     7849624
47097744...
-----------
47105593624
```



Contracted form.

$\dot{7}\dot{8}\dot{4}\dot{9}\dot{6}\dot{2}\dot{4} \times 6001$

47105593624

Wording in full: **4′**; **2′**; **6′**; 6 times $\dot{4}$ are 24, and $\dot{9}$ are **33′**, carry 3; 6 times $\dot{2}$ are 12, and 3 are 15, and $\dot{4}$ are **19′**, carry 1, &c.

Wording to be used: **4′**; **2′**; **6′**; 24, **33′**, carry 3; 12, 15, **19′**, carry 1; 36, 37, **45′**, carry 4; 54, 58, **65′**, carry 6; 24, **30′**, carry 3; 48, **51′**, carry 5; 42, **47′**

EXERCISE XXX.

(1) 35943628 × 201
(2) 9016706 × 301
(3) 11043690 × 401
(4) 763005090 × 501
(5) 2109863 × 6001
(6) 2109863 × 601
(7) 2109863 × 61
(8) 21098630 × 60001
(9) 475328956 × 7010
(10) 325408600 × 8010
(11) 325408600 × 810
(12) 325408600 × 180
(13) 104578000 × 901
(14) 104578000 × 1090
(15) 478006043 × 17
(16) 478006043 × 71
(17) 478006043 × 701
(18) 478006043 × 107
(19) 478006043 × 1007
(20) 478006043 × 7001

7. Multiply 48967 by 742.


```
   48967
     742
 -------
342769.. A
 2056614 B
 -------
36333514
```

Line A is 7 times the multiplicand; line B is 6 times line A, and being placed two figures to the right of line A, gives to the line A its required value, viz. 700 times the multiplicand.

Multiply 47213 by 568.


```
   47213
     568
 -------
  377704 A
 2643928  B
 -------
26816984
```

Line A is 8 times the multiplicand; line B is 7 times line A, or 56 times the multiplicand, and being written one place to the left, has the required value, viz. 560 times.

Similarly we can multiply by 1768 or 6817 in two lines, by noticing that $68 = 4 \times 17$. Multiply by 17 in one line, as in § 3.

This process may be still further contracted by adding the figures of the second product as fast as they are obtained to those of the first product.

```
47213 × 568
 ------
 377704
 ------
26816984
```

Wording of last line: 4′; 28′, carry 2; 0, 2, 9′, 49, 56′, carry 5; 49, 54, 61′, carry 6; 49, 55, 58′, carry 5; 21, 26′.

Exercise XXXI.

(1) 48927653 × 742
(2) 48927653 × 427
(3) 1093856 × 545
(4) 3256740 × 5450
(5) 18094300 × 8240
(6) 4760095 × 248
(7) 37823500 × 76300
(8) 42050000 × 6370
(9) 517328645 × 1872
(10) 4072908 × 5614
(11) 15827632 × 780130
(12) 243598000 × 1909500
(13) 357823492 × 50350
(14) 4768923150 × 720900
(15) 329616000 × 17068
(16) 52631578967 × 680017
(17) 89777630050 × 816
(18) 57234000905 × 954
(19) 853011009375 × 107214
(20) 46371968750 × 16480

8. We must now once more revert to Subtraction, and adopt another mode of reasoning and another form of words.*

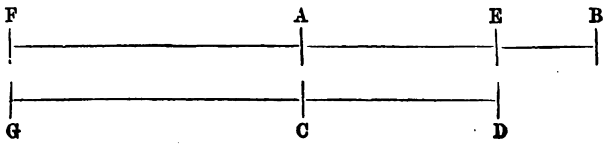
Find the difference between 430172 and 189567.

THE DIFFERENCE BETWEEN TWO NUMBERS WILL NOT BE ALTERED IF THE SAME QUANTITY IS ADDED TO BOTH OF THEM.

Illustrations: "Conceive two baskets with pebbles in them, in the first of which are 100 pebbles more than in the second. If I put 50 more pebbles into each of them, there are still only 100 more in the first than in the second."—*De Morgan's Arithmetic.*

The difference in age between a parent and his child will remain the same through life.

The difference, E B, between the lines A B and C D is not altered if A B and C D are equally lengthened to F and G.



CM.	XM.	M.	C.	X.	I.		
	10	10	10		10		
4	3	0	1	7	2	8	Minuend
1	8	9	5	6	7	0	Subtrahend
1	1	1		1			
2	4	0	6	0	5	8′	Difference

* It was not without some regret and much reflection that we did not at once in Chapter IV. adopt the method here given. An experience of upwards of twenty years has convinced us that the rationale of this process cannot be properly realized at the early stage at which Subtraction is necessary.

7 units cannot be taken from 2 units; we may add ten units to the minuend, provided we add the same, or its equivalent, 1 ten, to the subtrahend. Do so; and we have now to take 7 units from 12 units, leaving 5 units. In the tens' column we have now to subtract, not 6, but 7 tens; 7 tens from 7 tens leaves 0 tens; 5 hundreds, again, cannot be taken from 1 hundred; add 10 hundreds to the minuend, and its equivalent, 1 thousand, to the subtrahend. 5 hundreds from 11 hundreds leave 6 hundreds. Again, 10 thousands (not 9 thousands) cannot be taken from 0 thousands; add 10 thousands to the minuend, and its equivalent, 1 ten-thousand, to the subtrahend; 10 thousands from 10 thousands leave 0 thousands. Again, 9 ten-thousands (not 8) cannot be taken from 3 ten-thousands; add 10 ten-thousands to the minuend, and its equivalent, 1 hundred-thousand to the subtrahend; 9 ten-thousands from 13 ten-thousands leave 4 ten-thousands. 2 hundred-thousands not 1) from 4 hundred-thousands leave 2 hundred-thousands.

Find the difference between £502. 1*s.* 2*d.* and £138. 17*s.* 8¼*d.*

£.			£½.	*s.*	*d.*		
C.	X.	I.					
	10	10		20	12	$\frac{4}{4}$	
5	0	2		1	2		Minuend
1	3	8	1	7	8	$\frac{1}{4}$	Subtrahend
1	1	1		1	1		
3	6	3		3	5	$\frac{3}{4}$	Difference

1 farthing cannot be taken from 0; add 4 farthings to the minuend and 1 penny to the subtrahend; now 1 farthing from 4 farthings leaves 3 farthings. 9 (not 8) pence from 2 pence, we cannot; add 12 pence to the minuend and 1 shilling to the subtrahend; 9 from 14, 5 pence; 18 (not 17) from 1, we cannot; add 20 shillings to the minuend and £1 to the subtrahend; 18 from 21, 3 shillings. £9 (not 8) from £2 we cannot; add £10 to the minuend and 1 ten-pound note to the subtrahend; 9 from 12 leaves £3. Similarly 4 ten-pound notes from 10 ten-pound notes leave 6 ten-pound notes; 2 hundreds from 5 hundreds leave 3 hundreds.

9. The question, "7 from 9 leaves *what?*" is equivalent to the question, "7 and *what* makes 9?" *Answer.* 7 and $2' = 9$. For the *processes* we have in view this mode of expression must be adopted.

Subtract 8107214 from 9689476.

9689476	4
8107214	5
1582262	8′

Wording: 4 *and* **2′** are 6; *and* **6′** are 7; 2 *and* **2′** are 4; 7 *and* **2′** are 9; 0 *and* ′ are 8; 1 *and* **5′** are 6; 8 *and* **1′** are 9.

N.B. The figures after "and" must be written down *in the act of utterance.*

Subtract 176896 from 3401023.

3401023	4
176896	1
3224127	3′

Wording: 6 *and* **7′** are 13, carry 1; 10 *and* **2′** are 12, carry 1; 9 *and* **1′** are 10, carry 1; 7 *and* **4′** are 11, carry 1; 8 *and* **2′** are 10, carry 1; 2 *and* **2′** are 4; **3′**.

Exercise XXXII.

Work Exercise VII., using this wording.

10. Instead of uniformly adding 10 to both minuend and subtrahend, we might add any other number we pleased.

Subtract 1879642 from 3002161.

M^2.	CM.	XM.	M.	C.	X.	I.	
	10	20	80	50	40	30	
3	0	0	2	1	6	1	Minuend
1	8	7	9	6	4	2	Subtrahend
1	2	8	5	4	3		
1	0	5	68	41	39	29	

This answer must be correct, as we have committed no error in the reasoning; but, as it is inconvenient, we must alter its form, converting the 29 units into tens, the tens into hundreds, and so on.

29 units = **9′** units and 2 tens, which with the 39 tens = 41 tens = **1′** ten and 4 hundreds; (41 + 4) hundreds = 45 hundreds = **5′** hundreds and 4 thousands; (68 + 4) thousands = 72 thousands = **2′** thousands and 7 ten-thousands; (7 + 5) ten-thousands = 12 ten-thousands = **2′** ten-thousands and 1 hundred-thousand; (1 + 0) hundred-thousand = **1′** hundred-thousand, and there is **1′** million. *Answer.* 1122519.

The same answer will be obtained by the usual method.

3002161
1879642
1122519

11. Subtract the sum of 683, 1279, 43080, 1241596, 386724, and 427689, from 32684215 in one operation.

32684215	4	Minuend
683	8	Addenda and Subtrahend.
1279	1	
43080	6	
1241596	1	
386724	3	
427689	0	
30583164	3′	

The units of the subtrahend amount to 31, which cannot be taken from the 5 of the minuend; the most convenient number to be added to both minuend and subtrahend is evidently 30 units and 3 tens respectively. 31 and 4′ are 35; we therefore put down 4 and add or carry 3 to the next column of the subtrahend, and so on.

Wording: 9, 13, 19, 28, 31, *and* 4′ are 35, carry 3; 11, 13, 22, 30, 37, 45, *and* 6′ are 51, carry 5; 11, 18, 23, 25, 31, *and* 1′ are 32, carry 3; 10, 16, 17, 20, 21, *and* 3′ are 24, carry 2; 4, 12, 16, 20, *and* 8′ are 28, carry 2; 6, 9, 11, *and* 5′ are 16, carry 1; 2 *and* 0′ are 2; 3′.

EXERCISE XXXIII.

(1) From 5019308 take
62412
127842
5708
13052
58009
417925

(2) From 3785926 take
408620
7053
12019
38727
1968
423016

(3) From 348009052 take
7016094
14802060
24768923
5600082
28004015
13017019

(4) From 18050000 take
857142
857142
857142
857142
857142

(5) From 7009013 take
68459
194360
4011318
604098
28943
173666

(6) From 53685947 take
6078254
6078254
6078254
6078254
6078254
6078254

(7) From 5000000 take
18049
7683
354975
867324
841825
1019078

(8) From 10000000 take
345978
345978
345978
345978
345978
345978
345978
345978

(9) From 62947 take
6083
7509
1234
8765
9012
10010

(10) From 17685924 take
946877
946877
946877
946877
946877
946877
946877
946877
946877

12. Subtract 7 × 35698 from 310049.

310049	Minuend
35698 × 7	Subtrahend
60163	

Wording: 56 *and* 3′, 59, carry 5; 63, 68, *and* 6′, 74, carry 7; 42, 49, *and* 1′, 50, carry 5; 35, 40, *and* 0, 40, carry 4; 21, 25, *and* 6′, 31, carry 3; 3 *and* 0, 3.

Exercise XXXIV.

(1) 2530168 — 137519 × 7
(2) 40097123 — 240368 × 2
(3) 711160358 — 286709 × 9
(4) 4986342 — 111111 × 8
(5) 1000000 — 142857 × 7
(6) 6666329 — 700000 × 3
(7) 9876543210 — 1234567890 × 8
(8) 387640 — 28402 × 4
(9) 7628775 — 325755 × 5
(10) 705307060 — 38042016 × 6
(11) 494821309 — 43756801 × 9
(12) 980598 — 142857 × 6

13. Divide 478143296 by 26478.

Full form.

```
26478) 478143296 (18058
       26478
       213363
       211824
         153929
         132390
          215396
          211824
            3572
```

Contracted form.

```
26478) 478143296 (18058
       213363
         153929
          215396
            3572
```

Wording: 1st remainder—8 *and* 6′, 14, carry 1 ; 8 *and* 3′, 11, carry 1 ; 5 *and* 3′, 8 ; 6 *and* 1′, 7 ; 2 *and* 2′, 4. 2nd remainder—64 *and* 9′, 73, carry 7 ; 56, 63, *and* 3′, 66, carry 6 ; 32, 38, *and* 5′, 43, carry 4 ; 48, 52, *and* 1′, 53, carry 5 ; 16, 21. 3rd remainder—40 *and* ′, 49, carry 4 ; 35, 39, *and* 3′, 42, carry 4 ; 20, 24, *and* 5′, 29, carry 2 ; 30, 32, *and* 1, 33, carry 3 ; 10, 13, *and* 2′, 15. 4th remainder—64 *and* 2′, 66, carry 6 ; 56, 62, *and* 7′, 69, carry 6 ; 32, 38, *and* 5′, 43, carry 4 ; 48, 52, *and* 3′, 55, carry 5 ; 16, 21.

At this stage, if not sooner, the student will be able to guess at each figure of the quotient by mere inspection of the figures of highest denomination in the divisor and the successive dividends *without a table.* The first figure 1 is obvious, for though 2 is contained twice in 4, 26 is contained only once in 47. For the second figure, 26 in 213, 8 times and so on.

Divide £19043. 13*s.* 5¾*d.* by 415.

Contracted form.

```
       £.     s.  d.  £.   s.  d.
415) 19043   13  5¾ (45   17   9
      2443
       368
      ----
       737
      3223
       318
      ----
      3821
        86
      ----
       347 farthings over.
```

EXERCISE XXXV.

(1) 3097612 ÷ 8415
(2) 15000069 ÷ 9648
(3) £58943. 15*s.* 7¾*d.* ÷ 538
(4) £73914. 10*s.* 8½*d.* ÷ 2047
(5) 328016000 ÷ 20849
(6) 495627844 ÷ 16759
(7) £438629. 12*s.* 9½*d.* ÷ 60043
(8) 7012017 ÷ 638
(9) 35105090 ÷ 27643
(10) 459643729 ÷ 390458
(11) 326667052 ÷ 72046
(12) 2150842030246 ÷ 537689
(13) £457963824. 17*s.* 3¼*d.* ÷ 67902
(14) 1000000000 ÷ 53928
(15) 1000000 ÷ 13
(16) 15263318832 ÷ 2616
(17) 15263318832 ÷ 5834602
(18) £41265. 1*s.* 1½*d.* ÷ 5049
(19) £509446. 8*s.* 9*d.* ÷ 10010
(20) 2911902 ÷ 178

14. Multiply 358967 by 998.

Full form.

```
  358967
     998
 -------
 2871736
3230703
3230703
---------
358249066
```

1st contraction.

$358967 \times 998 = 358967 \times 1000 - 358967 \times 2$

```
358967000
   717934
---------
358249066
```

2nd contraction.

```
358967... × 998
---------
358249066
```

Wording: 14 and 6′, 20, carry 2; 12, 14, *and* 6′, 20, carry 2; 18, 20, *and* 0′, 20, carry 2; 16, 18, *and* 9′, 27, carry 2; 10, 12, *and* 4′, 16, carry 1; 6, 7, *and* 2′, 9; 8′; 5; 3′.

N.B. Special care must be taken in all these contractions to work with neatness, placing the results under the corresponding figures of the subtrahend.

Exercise XXXVI.

(1) 378645916 × 97	(5) 604580900 × 9200
(2) 14790804 × 996	(6) 748900 × 910
(3) 3017682 × 9993	(7) 208712 × 93 × 95
(4) 246060 × 95	(8) 267812479 × 9999

15. Divide 848008 by 56. According to the two interpretations of Division given in Chapter VIII. § 1, this may mean, either (*a*) "How many times is 56 contained in 848008?" or (*b*) "Distribute 848008 into 56 equal parts."

(*a*) Since $56 = 7 \times 8 = 8 \times 7$, and we have to group the 848008 things (say marbles) into lots of 56 marbles, we may first group them into lots of 8 marbles, and gather these lots 7 at a time, or first group them into lots of 7 marbles, and gather these lots 8 at a time.

8)848008 marbles
7)106001 lots of 8 marbles
15143 lots of 7 × 8 or 56 marbles.

7)848008 marbles
8)121144 lots of 7 marbles
15143 lots of 8 × 7 or 56 marbles.

(*b*) We have to distribute 848008 marbles among 56 persons, or among 7 companies of 8 persons, or among 8 companies of 7 persons.

7)848008 marbles
8)121144 marbles to each company of 8 persons
15143 marbles to each of these persons.

8)848008 marbles
7)106001 marbles to each company of 7 persons
15143 marbles to each person.

Divide 45648 by 36. $36 = 6 \times 6 = 4 \times 9 = 9 \times 4 = 3 \times 12 = 12 \times 3$.

6)45648	9)45648	4)45648	12)45648	3)45648
6)7608	4)5072	9)11412	3)3804	12)15216
1268	1268	1268	1268	1268

$36 = 2 \times 18$, but this resolution of 36 into factors is not available, because we only know the multiplication table up to twelves. We have thus the choice of the five different ways here given.

Divide £5159. 1*s.* 3*d.* among 45 persons.

	£.	*s.*	*d.*	
5)	5159	1	3	
9)	1031	16	3	to each 9
	114	12	11	to each 1

	£.	*s.*	*d.*	
9)	5159	1	3	
5)	573	4	7	to each 5
	114	12	11	to each 1

Exercise XXXVII.

(1) 261912 ÷ 56
(2) £11537. 6*s.* $4\frac{1}{4}$*d.* ÷ 25
(3) 205515 ÷ 45
(4) £901. 1*s.* ÷ 36
(5) 76624 ÷ 16
(6) £1270. 10*s.* ÷ 32
(7) 593075 ÷ 35
(8) £62578. 9*s.* $9\frac{3}{4}$*d.* ÷ 81
(9) 512784 ÷ 144
(10) £17075. 11*s.* $2\frac{1}{4}$*d.* ÷ 63
(11) 7777728 ÷ 63
(12) £16967. 14*s.* ÷ 144

N.B. In this Exercise the interpretation of each line must be given.

16. Divide 438000 by 10. This means, How many times is 10 contained in 438000? or, How many tens are there in 438000? *Ans.* 43800. (See Ch. I. § 11.)

Divide 3745916 by 100. *Ans.* 37459 and 16 over.

Divide 42097412 by 10000. *Ans.* 4209 and 7412 over; hence,

Learn by heart: *To divide by any power of* 10, *strike off from the right of the dividend as many figures as there are ciphers in the divisor.* This is evidently the converse of the rule in Ch. VI. § 6.

Divide £23519. 12*s.* 8½*d.* by 10. Dividing the pounds, we obtain £2351 and £9 over; £9 and 12 shillings = 192 shillings, giving 19 shillings and 2 shillings over; 2 shillings and 8 pence = 32 pence, giving 3 pence and 2 pence over, which with the halfpenny make 10 farthings, giving 1 farthing.

	£.	s.	d.
10)	23519	12	8½
	2351	19	3¼

Divide £746915. 6*s.* 2½*d.* by 1000.

	£.	s.	d.
1000)	746(915	6	2½
	18(306		
	3(674		
	2(698		

Ans. £746. 18*s.* 3½*d.* and 698 farthings over.

N.B. We forbear giving the name to the result, as the "numerical values of the quotient and remainder are the same" under each interpretation. (See Ch. VIII. § 7.)

Exercise XXXVIII.

(1) 3257968 ÷ 10, 100, 1000, 10000.

(2) £8684. 7*s.* 6*d.* ÷ 10, 100, 1000, 10000.

(3) £39791. 13*s.* 4*d.* ÷ 10, 100, 1000, 10000, 100000.

(4) £243519. 16*s.* 5*d.* ÷ 1000.

(5) £4000019. 6*s.* ÷ 10000.

(6) £59. 9*s.* 7*d.* ÷ 100.

17. Divide 4321232 by 56.

Under interpretation (*a*).

```
8)4321232
 7)540154 eights
    77164 fifty-sixes and 6 eights over.
```

Remainder, $6 \times 8 = 48$.

or,

```
7)4321232
 8)617318 sevens and 6 units over
    77164 fifty-sixes and 6 sevens over.
```

Remainder, $6 \times 7 + 6 = 48$.

Under interpretation (*b*).

```
8)4321232
 7)540154 to each of the 8 groups
    77164 to each one, and 6 over from each group.
```

Remainder, $6 \times 8 = 48$.

```
7)4321232
 8)617318 to each of the 7 groups, and 6 over
    77164 to each one, and 6 over from each group.
```

Remainder, $6 \times 7 + 6 = 48$.

£37591. 15*s.* $7\frac{1}{4}$*d.* $\div$ 45.

```
5)37591 15 7¼
 9)7518  7 1¼ to each of the 5 sets, and 4 farthings over
    835  7 5¼ to each one, and 8 farthings over from each set.
```

Total remainder, $5 \times 8 + 4 = 44$ farthings.

or,

```
9)37591 15 7¼
 5)4176 17 3¼ to each of the 9 groups, and 8 farthings over
    835  7 5¼ and 4 farthings over from each group.
```

Total remainder, $4 \times 9 + 8 = 44$ farthings.

Exercise XXXIX.

(1) $458239 \div 14$

(2) £670138. 12*s.* 11*d.* $\div$ 66

(3) $826549 \div 45$

(4) £764. 18*s.* $2\frac{3}{4}$*d.* $\div$ 72

(5) $2761324 \div 81$

(6) £5374. 17*s.* 6*d.* $\div$ 15

(7) $14685999 \div 49$

(8) £27632. 4*s.* $3\frac{1}{2}$*d.* $\div$ 64

(9) $7632419 \div 77$

(10) £1000000 $\div$ 84

(11) $5768341 \div 132$

(12) £16349. 12*s.* 7*d.* $\div$ 42

18. 374916 ÷ 70.

```
10)37491(6
  7)37491  and 6 over
    5355  and 6×10+6=66 over.
```

or, in one line,

```
70)37491(6
    5355   and 66 over.
```

437492563 ÷ 8000.

```
8000)437492(563
      54686 and 4563 over.
```

58317409 ÷ 359000.

```
359000)58317(409 (162
        2241
          877
          159409        Ans. 162, and 159409 over.
```

Exercise XL.

(1) 315672 ÷ 20
(2) 8409136 ÷ 30
(3) 8409136 ÷ 40
(4) 8409136 ÷ 50
(5) 437589601 ÷ 90
(6) 8392 ÷ 60
(7) 595536900 ÷ 70
(8) 2359360 ÷ 80
(9) 904813 ÷ 600
(10) 5897343 ÷ 5000
(11) 388493200 ÷ 9000000
(12) 3596250000 ÷ 5000
(13) 57892517 ÷ 63200
(14) 34603421 ÷ 7410000
(15) 368254000 ÷ 5300000
(16) 1498632000 ÷ 730
(17) 7765450000 ÷ 38500
(18) 36932215800000 ÷ 738600
(19) 487563625 ÷ 50
(20) 487563625 ÷ 500
(21) 487563625 ÷ 5000
(22) 487563625 ÷ 50000
(23) 4875636250 ÷ 50000
(24) 48756362500 ÷ 50000
(25) 487563625000 ÷ 50000

19. 5 = 10 ÷ 2
25 = 100 ÷ 4
75 = 300 ÷ 4
125 = 1000 ÷ 8
375 = 3000 ÷ 8
625 = 5000 ÷ 8
875 = 7000 ÷ 8

Hence to multiply by 5, we may multiply by 10 and divide by 2.

„	25,	„	100	„	4.
„	75,	„	300	„	4.
„	125,	„	1000	„	8.
„	375,	„	3000	„	8.
„	625,	„	5000	„	8.
„	875,	„	7000	„	8.

Multiply 437911 by 25.

$$4)\underline{43791100}$$
$$10947775 \qquad \textit{Ans.}\ 10947775.$$

Multiply 8923761 by 375.

$$\begin{array}{r} 8923761 \\ \underline{3} \\ 8)\underline{26771283000} \\ 3346410375 \end{array} \qquad \textit{Ans.}\ 3346410375.$$

EXERCISE XLI.

(1) 381961 × 25
(2) 621852 × 75
(3) 776877 × 125
(4) 492743 × 375
(5) 276768 × 625
(6) 512634 × 875
(7) 85059 × 125
(8) 52580 × 25
(9) 385149 × 625
(10) 103416 × 75
(11) 94238 × 375
(12) 723498 × 875
(13) 427 × 25 × 25 × 25
(14) 1853 × 125 × 25 × 875
(15) 1006 × 375 × 5 × 75 × 1000
(16) 512 × 125 × 375 × 625 × 875

CHAPTER X.

SCALES OF NOTATION.

1. A Scale of Notation may be simple or compound.

In a simple scale, such as those named in Ch. I. § 10, we pass from one column to the next higher one by the *same* multiplier or *radix*. Thus in the Binary scale the radix is 2, in the Ternary 3, in the Decimal 10, and so on.

In a compound scale the radix changes; thus in the Money scale it is successively 4, 12, 10, 2, and then permanently 10. In Avoirdupois Weight it is 16, 16, 28, 4, 20, and again permanently 10, and so on.

2. SIMPLE SCALES. Addition.

Add the following quantities, expressed in the quinary scale: 2041, 41012, 12340, 123, 333, 33404.

N.B. 5×5 is written 5^2, $5\times5\times5$ is written 5^3, $5\times5\times5\times5$ is written 5^4, and so on.

5^4	5^3	5^2	5	1	
	2	0	4	1	3 *
4	1	0	1	2	0
1	2	3	4	0	2
		1	2	3	2
		3	3	3	1
3	3	4	0	4	2
2 0	0	4	1	3	2′

Since 5 in any column gives 1 in the column next to the left, we add each column separately, carrying every 5 as 1 to the next column, beginning (as in ordinary addition) on the right. (Ch. III. § 4.)

Wording: 4, 7, 10, 12, 13, 3′, carry 2; 5, 7, 11, 12, 16, 1′, carry 3; 7, 10, 11, 14, 4′, carry 2; 5, 7, 8, 10, 0′, carry 2; 5, 6, 10, 0′; 2′. *Answer.* 200413. (This must be read, "two, nought, nought, four, one, three.")

Exercise XLII.

(1) Add in the binary scale, 111, 1100, 1101, 1110, 11001, 1001, 11101.

(2) In the ternary scale, 1220, 2012, 2111, 210, 12112, 222, 1221.

(3) In the quaternary scale, 1032, 1222, 22321, 1211, 1002, 12223, 3232.

(4) In the quinary scale, 2341, 1234, 110, 2323, 443322, 12340, 342103.

(5) In the senary scale, 54320, 234, 5030, 24110, 25142, 33445, 55443.

(6) In the septenary scale, 6543, 6321, 1324, 235, 35264, 10235.

(7) In the octonary scale, 76321, 4623, 5276, 35402, 70607, 42354.

(8) In the nonary scale, 1235, 7834, 72684, 503785, 123456, 78064.

(9) In the undecimal scale (using *t* for ten), 6*t*432, 12579, 708*t*4, 5*tt*37*t*.

(10) In the duodecimal scale (using *t* for ten and *e* for eleven) 6*te*4*e* 7*eet*4, 897*e*5, 3364*t*, *tetet*, *eeeee*.

* Casting out fours, 4 being 1 less than the radix, just as 9 is 1 less than the radix in the decimal scale. (Cf. Ch. XI. § 12.)

(11) In each of the preceding scales, 11111 + 10101 + 11001 + 10011 + 11001 + 10111 + 1111 + 1001 + 11001 + 1 + 111 + 101.

3. Subtraction.

In the senary scale, take 45305 from 305213.

$$
\begin{array}{cccccc|c}
6^5 & 6^4 & 6^3 & 6^2 & 6 & 1 & \\
\hline
3 & 0 & 5 & 2 & 1 & 3 & 4 \\
 & 4 & 5 & 3 & 0 & 5 & 2 \\
\hline
2 & 1 & 5 & 5 & 0 & 4 & 2'
\end{array}
$$

Here if any figure of the subtrahend is greater than the figure of the minuend above it, we add 6 to the figure of the minuend, carrying 1 to the next column to the left. (Ch. IX. p. 103.)

Wording: **5 and 4′ is 9, carry 1; 1 and 0′ is 1; 3 and 5′ is 8, carry 1; 6 and 5′ is 11, carry 1; 5 and 1′ is 6, carry 1; 1 and 2′ is 3.**

EXERCISE XLIII.

(1) Take in the binary scale, 101010 from 1110101.

(2) In the ternary scale, 211021 from 1002101.

(3) In the quaternary scale, 121323 from 303030.

(4) In the quinary scale, 32402 from 40000.

(5) In the senary scale, 12345 from 323520.

(6) In the septenary scale, 135066 from 423450.

(7) In the octonary scale, 3407427 from 7262520.

(8) In the nonary scale, 326784 from 2233441.

(9) In the undecimal scale, 7*t*4*t*3 from 102030.

(10) In the duodecimal scale, 7*teet*407 from 89*t*4007*te*.

(11) In each of the preceding scales, 101101011 from 110010010.

4. Multiplication.

42345 × 3 in the senary scale.

$$
\begin{array}{cccccc}
6^5 & 6^4 & 6^3 & 6^2 & 6 & 1 \\
\hline
 & 4 & 2 & 3 & 4 & 5 \\
 & & & & & 3 \\
\hline
2 & 1 & 1 & 5 & 2 & 3
\end{array}
$$

Wording: 15, 3′, carry 2; 12, 14, 2′, carry 2; 9, 11, 5′, carry 1; 6, 7, 1′, carry 1; 12, 13, 1′; 2′.

Ans. 211523.

64325 × seven, in the septenary scale.

	7^4	7^3	7^2	7	1
	6	4	3	2	5
			seven	= 1	0
6	4	3	2	5	0

Wording: 35, 0′, carry 5; 14, 19, 5′, carry 2; 21, 23, 2′, carry 3; 28, 31, 3′, carry 4; 42, 46, 4′; 6′.

Comparing the product with the multiplicand, we see that a cipher has been added in the units' place, the other figures remaining unchanged. This might have been anticipated, for the cipher in the units' place moves each figure one step higher in the septenary scale. (Cf. Ch. V. p. 53.)

Note that in any scale the symbol for the radix is 10, and for the successive powers of the radix 100, 1000, &c.

Generally: To multiply in any scale by a power of the radix, put on as many ciphers to the right of the multiplicand as there are ciphers in the multiplier expressed in that scale. (Cf. Ch. VI. p. 66.)

462037 × 2056 in the octonary scale.

Casting out sevens:

$$\begin{array}{ccc} & 6 & \\ 1 & \times & 6 \\ & 6 & \end{array}$$

8^9	8^8	8^7	8^6	8^5	8^4	8^3	8^2	8	1
				4	6	2	0	3	7
						2	0	5	6
			3	4	5	4	2	7	2
		2	7	7	2	2	3	3	
1	1	4	4	0	7	6			
1	1	7	7	4	7	4	6	2	2

Ans. 1177474622.

te3t7 × *t3e8* in the duodecimal scale.

Casting out elevens:

$$\begin{array}{ccc} & 3 & \\ 8 & \times & t \\ & 3 & \end{array}$$

12^8	12^7	12^6	12^5	12^4	12^3	12^2	12	1
				t	*e*	3	*t*	7
					t	3	*e*	8
			7	3	6	7	0	8
		t	0	4	6	8	5	
	2	8	9	*e*	7	9		
9	1	5	2	9	*t*			
9	5	0	8	5	7	0	5	8

Ans. 950857058.

EXERCISE XLIV.

(1) 1011101 × 10011 in the binary scale.
(2) 2122002 × 1212 in the ternary scale.
(3) 3130321 × 30012 in the quaternary scale.
(4) 123404 × 3204 in the quinary scale.
(5) 53210 × 1305 in the senary scale.
(6) 130666 × 651 in the septenary scale.
(7) 46734 × 730 in the octonary scale.
(8) 80600 × 71000 in the nonary scale.
(9) 457*t* × 4*tt*1 in the undecimal scale.
(10) 6*et*1 × 6*et*1 in the duodecimal scale.
(11) 1100 × 1000 in every scale.

5. Division.

7341062 ÷ 5 in the octonary scale.

```
5)7341062
  1371643—3
```

Wording: 5 in 7, 1′, carry 2; (2 × 8 + 3 = 19) in 19, 3′, carry 4, 32; in 36, 7′, carry 1, 8; in 9, 1′, carry 4, 32; in 32, 6′, carry 2, 16; in 22, 4′, carry 2, 16; in 18, 3′, and 3 over.

Note that to divide by a power of the radix, we cut off as many figures from the right of the dividend as there are ciphers in the divisor, expressed in that scale.

EXERCISE XLV.

(1) 1000110 ÷ 1000 in the binary scale.
(2) 12112 ÷ 2 in the ternary scale.
(3) 321232 ÷ 3 in the quaternary scale.
(4) 100342 ÷ 3 in the quinary scale.
(5) 132432 ÷ seven in the quinary scale.
(6) 543212 ÷ 4 in the senary scale.
(7) 5372461 ÷ 7 in the octonary scale.
(8) 33776426 ÷ 6 in the nonary scale.
(9) 123456 ÷ *t* in the undecimal scale.
(10) 5732*te*2 ÷ *t* in the duodecimal scale.

N.B. These scales are of little practical utility, and we therefore omit Long Division.

6. Interconversion of simple scales.

Express 437 (decimal) in the quinary scale.

$$\begin{array}{rl} & \text{C. X. I.} \\ 5) & \underline{4\ 3\ 7} \\ 5) & \underline{8\ 7} \text{ fives and 2 units} \\ 5) & \underline{1\ 7} \text{ twenty-fives and 2 fives} \\ & 3 \text{ hundred-and-twenty-fives and 2 twenty-fives} \end{array}$$

Hence 437 (decimal) = 3222 (quinary).

Convert 73421 from the octonary to the ternary scale.

$$\begin{array}{rl} & \underline{8^4\ 8^3\ 8^2\ 8\ 1} \\ 3) & \underline{7\ 3\ 4\ 2\ 1} \\ & 2\ 3\ 6\ 6\ 0 \text{ and 1 unit over} \end{array}$$

$\therefore$ 73421 in the octonary scale $= 23660 \times 3$ still expressed in the octonary scale and 1 unit over.

$$\begin{array}{rl} 3) & \underline{23660} \\ & 6472 \text{ and 2 over} \end{array}$$

$\therefore$ 73421 in the octonary scale $= 6472 \times 3^2 + 2 \times 3 + 1$.

$$\begin{array}{rl} 3) & \underline{6472} \\ & 2150 \text{ and 2 over} \end{array}$$

$\therefore 73421 = 2150 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 1.$

$$\begin{array}{rl} 3) & \underline{2150} \\ & 570 \text{ and 0 over} \end{array}$$

$\therefore 73421 = 570 \times 3^4 + 0 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 1.$

$$\begin{array}{rl} 3) & \underline{570} \\ & 175 \text{ and 1 over} \end{array}$$

$\therefore 73421 = 175 \times 3^5 + 1 \times 3^4 + 0 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 1.$

$$\begin{array}{rl} 3) & \underline{175} \\ & 51 \text{ and 2 over} \end{array}$$

$\therefore 73421 = 51 \times 3^6 + 2 \times 3^5 + 1 \times 3^4 + 0 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 1.$

$$\begin{array}{rl} 3) & \underline{51} \\ & 15 \text{ and 2 over} \end{array}$$

$\therefore 73421 = 15 \times 3^7 + 2 \times 3^6 + 2 \times 3^5 + 1 \times 3^4 + 0 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 1.$

$$\begin{array}{rl} 3) & \underline{15} \\ & 4 \text{ and 1 over} \end{array}$$

$\therefore 73421 = 4 \times 3^8 + 1 \times 3^7 + 2 \times 3^6 + 2 \times 3^5 + 1 \times 3^4 + 0 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 1.$

$$\begin{array}{r} 3\underline{)4} \\ 1 \text{ and } 1 \text{ over} \end{array}$$

$\therefore\ 73421 = 1 \times 3^9 + 1 \times 3^8 + 1 \times 3^7 + 2 \times 3^6 + 2 \times 3^5 + 1 \times 3^4 + 0 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 1.$

$$\therefore \quad \frac{8^4\ 8^3\ 8^2\ 8\ \ 1}{7\ \ 3\ \ 4\ \ 2\ \ 1} = \frac{3^9\ 3^8\ 3^7\ 3^6\ 3^5\ 3^4\ 3^3\ 3^2\ 3\ \ 1}{1\ \ 1\ \ 1\ \ 2\ \ 2\ \ 1\ \ 0\ \ 2\ \ 2\ \ 1}$$

Mod. op.:

$$\begin{array}{r} 3\underline{)73421} \\ 3\underline{)23660}\text{—}1 \\ 3\underline{)6472}\text{—}2 \\ 3\underline{)2150}\text{—}2 \\ 3\underline{)570}\text{—}0 \\ 3\underline{)175}\text{—}1 \\ 3\underline{)51}\text{—}2 \\ 3\underline{)15}\text{—}2 \\ 3\underline{)4}\text{—}1 \\ 1\text{—}1 \end{array}$$

Ans. 1112210221.

In reducing to the decimal scale, the following method may be used with equal advantage :

Reduce 462351 from the septenary to the decimal scale.

$$\begin{array}{rl} & \underline{7^5\ 7^4\ 7^3\ 7^2\ 7\ \ 1} \\ & 4\ \ 6\ \ 2\ \ 3\ \ 5\ \ 1 \\ & \underline{7} \\ \text{(thirty-four)} & 34\ (7^4) \\ & \underline{7} \\ \text{(two hundred \& forty)} & 240\ (7^3) \\ & \underline{7} \\ \text{\&c.} & 1683\ (7^2) \\ & \underline{7} \\ & 11786\ (7) \\ & \underline{7} \\ & 82503 \text{ units} \end{array}$$

or, as before :

$$\begin{array}{r} \underline{7^5\ 7^4\ 7^3\ 7^2\ 7\ \ 1} \\ \text{ten}\underline{)\ 4\ \ 6\ \ 2\ \ 3\ \ 5\ \ 1} \\ \text{ten}\underline{)\ 3\ \ 3\ \ 0\ \ 2\ \ 4}\text{—}3 \\ \text{ten}\underline{)\ 2\ \ 2\ \ 5\ \ 6}\text{—}0 \\ \text{ten}\underline{)\ 1\ \ 4\ \ 5}\text{—}5 \\ \text{ten}\underline{)\ 1\ \ 1}\text{—}2 \\ 0\text{—}8 \end{array}$$

Ans. 82503.

4 of the 6th column gives $7 \times 4 = 28$ of the 5th column, which, with the 6 already in that column, gives 34. 34 of the 5th column *gives* 7×34 of the 4th column, and adding in the 2 gives 240, &c

Exercise XLVI.

(1) Express 1024* in the binary scale.

(2) Express 719 in the binary scale.

(3) Express 719 in the ternary scale.

(4) Express 1000 in the quaternary scale.

(5) Express 760 in each scale from the binary to the duodecimal scale.

(6) Express 20453 in the same scales.

(7) Convert 3742 (nonary) to the decimal scale.

(8) Convert 51342 (senary) to the decimal scale.

(9) 123402 (quinary) to the decimal scale.

(10) 45312 (septenary) to the decimal scale.

(11) 10111001 (binary) to the decimal scale.

(12) 4*eete*3 (duodecimal) to the decimal scale.

(13) Convert 11001 (binary) to the quinary scale.

(14) 31426 (octonary) to the septenary scale.

(15) 12121210 (ternary) to the quaternary scale.

(16) 5*t*31*t*9 (undecimal) to the nonary scale.

(17) *te*3418 (duodecimal) to the octonary scale.

(18) 100000000 (binary) to each of the other scales.

7. Compound Scales. These methods are applicable to the interconversion of £. *s.* *d.*, weights and measures of length, capacity, time, &c. We proceed to give examples of the leading weights and measures, but propose to enter more fully into the whole subject at a later stage.

Reduce 7419 farthings to £. *s.* *d.*

$$\begin{array}{rl} 4)\underline{7419} & \text{farthings} \\ 12)\underline{1854} & \text{pence and 3 farthings} \\ 20)\underline{154} & \text{shillings and 6 pence} \\ 7 & \text{pounds and 14 shillings.} \end{array}$$

Ans. £7. 14*s.* $6\frac{3}{4}$*d.*

Exercise XLVII.

(1) Reduce 1000000 farthings to £. *s.* *d.*

(2) Find the value of a million penny postage-stamps.

(3) Reduce 987654321 farthings to £. *s.* *d.*

* Where no scale is specified, the decimal scale is understood.

(4) Reduce 2479 sixpences to £. s. d.

(5) Reduce 2573 half-crowns to £. s. d. (N.B. 8 half-crowns = £1).

(6) Reduce 17019 fourpenny-pieces to £. s. d.

(7) Reduce to £. s. d. the following:

1. 43794 farthings.
2. 47901 farthings.
3. 637425 farthings.
4. 111840 farthings.
5. 197311 halfpence.
6. 16049 pence.

8. Lengths.

12 inches (in.)	=	1 foot (ft.)
3 feet	=	1 yard (yd.)
220 yards	=	1 furlong (fur.)
8 furlongs (1760 yards)	=	1 mile (m.)

Reduce 37989511 inches to miles, &c.

$$\begin{array}{rl}
12\underline{)37989511} & \text{inches} \\
3\underline{)3165792} & \text{feet and 7 inches} \\
220\left\{\begin{array}{r} 20\underline{)105526(4} \\ 11\underline{)52763} \end{array}\right. & \begin{array}{l}\text{yards and 0 feet} \\ \text{scores of yards and 4 yards} \end{array} \\
8\underline{)4796} & \text{furlongs and 7 scores of yards} \\
599 & \text{miles and 4 furlongs.}
\end{array}
\quad \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \begin{array}{l} 7\times 20+4 \\ =144 \text{ yards.} \end{array}$$

Ans. 599 miles, 4 furlongs, 144 yards (0 feet), 7 inches.

Exercise XLVIII.

(1) Reduce 79 inches to feet.

(2) „ 159 inches to yards, &c.

(3) „ 1000 inches to fathoms (1 fathom = 6 feet).

(4) „ 5000 yards to miles.

(5) 1. Reduce 5317 inches to yards.
2. „ 16029 „ „
3. „ 867 „ „
4. „ 1868 „ „
5. „ 4428 „ „
6. „ 2340 „ „

(6) 1. Reduce 176000 yards to miles.
2. „ 1000000 „ „
3. „ 74912 „ „
4. „ 16016 „ „

(7) Reduce 500497056 inches to miles, &c.

9.

100 links = 1 chain (22 yards)
80 chains = 1 mile

Exercise XLIX.

(1) Reduce 7843 links to chains.
(2) „ 53984 links to miles.
(3) „ 174986 links to miles.
(4) „ 1000000 links to miles.

10.

4 nails (n.) = 1 quarter
4 quarters = 1 yard

Exercise L.

(1) Reduce 243 nails to yards.
(2) „ 587 „ „
(3) „ 1024 „ „

11. Liquid Measure.

4 gills = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gn.)

Exercise LI.

(1) Reduce 5317 pints to gallons.
(2) „ 20000 gills to gallons.
(3) „ 3719 quarts to gallons.

12. Dry Measure.

2 gallons = 1 peck (pk.)
4 pecks = 1 bushel (bus.)
8 bushels = 1 quarter (qr.)

Exercise LII.

(1) Reduce 1700 gallons to quarters.
(2) „ 1359 gallons to quarters.
(3) „ 1000000 gallons to quarters.

13. Paper Measure.

24 sheets = 1 quire
20 quires = 1 ream

Exercise LIII.

(1) Reduce 1750 sheets to reams.
(2) „ 157 quires to reams.
(3) „ 1920 sheets to reams.

14. **Weights.**

1. *Avoirdupois.*

16 drams (dr.)	=	1 ounce (oz.)
16 ounces	=	1 pound (lb.)
28 lbs.	=	1 quarter (qr.)
4 qrs. (112 lbs.)	=	1 hundred-weight (cwt.)
20 cwts.	=	1 ton.

Exercise LIV.

Reduce:

(1) 1250 drams to ounces.
(2) 512 drams to ounces.
(3) 10000 drams to lbs.
(4) 2055 drams to lbs.
(5) 5040 lbs. to cwts.

Reduce:

(6) 11111 lbs. to cwts.
(7) 100000 lbs. to tons.
(8) 25200 lbs. to tons.
(9) 1000 cwts. to tons.
(10) 27419 lbs. to tons.

15. 2. *Troy.*

24 grains (gr.)	=	1 penny-weight (dwt.)
20 dwts.	=	1 ounce (oz.)
12 oz.	=	1 lb.

Exercise LV.

(1) Reduce 500 grains to dwts.
(2) „ 5760 grains to lbs.
(3) „ 7000 grains to lbs.
(4) „ 157409 grains to lbs.

N.B. Compare the results of (2) and (3). 7000 grains = 1 lb. av., which is therefore greater than 1 lb. troy. 1 oz. av., however, is less than 1 oz. troy.

16. Reduce 45 miles to feet.

$$\begin{array}{r r l}
 & 45 & \text{miles} \\
 & \underline{8} & \\
 & 360 & \text{furlongs} \\
220\left\{\right. & \underline{11} & \\
 & 3960 & \\
 & \underline{20} & \\
 & 79200 & \text{yards} \\
 & \underline{3} & \\
 & 237600 & \text{feet.}
\end{array}$$

Ans. 237600 ft.

Reduce 5 miles, 3 fur., 143 yds., 1 ft., to inches.

```
     5 m., 3 fur., 143 yds., 1 ft.
     8
    --
    43 fur.
    11
   ---
   473
    20
  ----
  9603 yds.
     3
 -----
 28810 ft.
    12
------
345720 inches.          Ans. 345720 in.
```

Reduce 4 tons, 13 cwt., 1 qr., 11 oz., to ounces.

```
     4 tons, 13 cwt., 1 qr. (0 lbs.), 11 oz.
    20
    --
    93 cwt.
     4
   ---
   373 qrs.
     4
  ----
  1492
     7
 -----
 10444 lbs.
    16
------
167115 ounces.          Ans. 167115 oz.
```

EXERCISE LVI.

(1) Reduce 7 yds., 2 ft., 9 in., to inches.
(2) „ 3 m., 1 fur., 93 yds., to feet.
(3) „ £73. 14s. $7\frac{1}{4}d.$ to farthings.
(4) „ £100 to sixpences.
(5) „ 6s. 8d. to fourpenny-pieces.
(6) „ 5 half-crowns to threepenny-pieces.
(7) „ £7. 17s. 6d. to sixpences.
(8) „ £99. 19s. $11\frac{3}{4}d.$ to farthings.
(9) „ £19. 15s. to crowns.
(10) „ £52. 10s. to half-crowns.
(11) „ £6. 13s. $7\frac{1}{2}d.$ to halfpence.
(12) „ £18. 2s. $9\frac{1}{2}d.$ to farthings.
(13) „ 100 tons to lbs.

(14) Reduce 8 cwt., 11 oz., to oz.
(15) „ 5 reams to sheets.
(16) „ 1 year to seconds.
(17) How many beats does the seconds' pendulum make in a week?
(18) How many miles can I travel for £3. 7*s.* 10*d.* at a penny a mile?

17. Addition, Subtraction, Multiplication and Division of Weights and Measures may be performed by methods similar to those already given for £. *s.* *d.*

(1) Add the following: 17 tons, 5 cwt., 1 qr., 13 lbs.; 25 tons, 3 cwt., 15 lbs.; 9 tons, 6 cwt., 1 qr., 21 lbs.; 14 cwt., 3 qrs., 11 lbs.; 73 tons, 17 lbs.; 1 ton, 13 cwt., 14 lbs.

Tons.	cwt.	qrs.	lbs.
17	5	1	13
25	3	0	15
9	6	1	21
	14	3	11
73	0	0	17
1	13	0	14
127	3	0	7

Wording: 4, 11, 12, 13, 18, 21, carry 2; 3, 4, 5, 7, 8, 9, 91 lbs. (28 in 91, 3 and 7 over), 7′, carry 3; 6, 7, 8, 0′, carry 2; 5, 9, 15, 18, 23′, carry 2; 3, 4, carry 2; 3, 6, 15, 20, 27′, carry 2; 9, 11, 12′.

Ans. 127 tons, 3 cwt., 7 lbs.

(2) From 19 tons, 5 cwt., 11 lbs., take 2 tons, 17 cwt., 1 qr., 27 lbs.

Tons.	cwt.	qrs.	lbs.
19	5	0	11
2	17	1	27
16	7	2	12

Wording: 27 and 12′ is 39 (adding 1 qr. = 28 lbs. to minuend and subtrahend), carry 1; 2 and 2′ is 4, carry 1; 18 and 7′ is 25, carry 1; 3 and 6′ is 9; 1′.

Ans. 16 tons, 7 cwt., 2 qrs., 12 lbs.

(3) 15 tons, 3 cwt., 1 qr., 13 lbs. × 4023.

	Tons.	cwt.	qrs.	lbs.
I.	15	3	1	13 × 3
X.	151	13	2	18 × 2
C.	1516	16	2	12
M.	15168	6	0	8 × 4
	60673	4	1	4
	303	7	1	8
	45	10	0	11
	61022	1	2	23

Ans. 61022 tons, 1 cwt., 2 qrs., 23 lbs.

(4) 251 tons, 3 cwt., 4 lbs. ÷ 11 tons, 5 cwt., 1 lb.

					11 tons, 5 cwt., 0 qrs., 1 lb.)251	3	0	4	X.I. (22
Tons.	cwt.	qrs.	lbs.		225	0	0	20	
11	5	0	1	I.	26	2	3	12	
112	10	0	10	X.	22	10	0	2	
					3	12	3	10	

Ans. 22 times and 3 tons, 12 cwt., 3 qrs., 10 lbs. over.

(5) 734 tons, 4 cwt., 1 qr., 12 lbs. ÷ 17.

Tons.	cwt.	qrs.	lbs.	Tons.	cwt.	qrs.	lbs.
17)734	4	1	12	(43	3	3	4
54							
3							
64							
13							
53							
2							
68							

Ans. 43 tons, 3 cwt., 3 qrs., 4 lbs.

Exercise LVII.

(1) Add 50 tons, 17 cwt., 3 qrs., 15 lbs.; 12 tons, 12 cwt., 12 lbs.; 25 tons, 11 cwt., 7 lbs.; 33 tons, 15 cwt., 2 qrs., 23 lbs.

(2) Add 5 oz., 13 dwt., 7 grs.; 11 oz., 10 dwts., 17 grs.; 2 oz., 15 grs.; 17 dwt., 14 grs.; 3 oz., 23 grs.

(3) 5 miles, 7 fur., 13 yds. + 19 miles, 1 fur., 200 yds. + 11 miles, 3 fur., 37 yds. + 2 fur., 183 yds. + 17 miles, 5 fur., 177 yds.

(4) 24 hrs., 11 min., 25 sec. + 19 hrs., 17 min., 11 sec. + 1 hr., 49 min., 50 sec. + 20 hrs., 20 min., 20 sec.

(5) 13 tons, 17 cwt., 13 lbs. — 4 tons., 15 cwt., 3 qrs., 20 lbs.

(6) 9 oz., 3 dwt., 3 grs. — 3 oz., 13 dwt., 7 grs.

(7) 20 miles — 11 miles, 1 fur., 17 yds.

(8) From 7 yds., 11 inches, take 3 yds., 2 feet, 7 inches.

(9) 8 tons, 11 cwt., 1 qr., 14 lbs. × 308.

(10) 5 oz., 5 dwt., 17 grs. × 3160.

(11) 5 miles, 155 yds. × 188.

(12) 237 tons, 18 cwt., 3 qrs., 20 lbs. ÷ 4 cwt., 3 qrs., 1 lb.

(13) 20 lbs. troy $\div$ 15 dwts., 11 grs.

(14) 7 tons, 11 cwt., 14 lbs. $\div$ 7.

(15) 13 lbs. troy $\div$ 32.

(16) 173 tons, 5 cwt., 1 qr., 3 lbs. $\div$ 19.

(17) 24 miles $\div$ 55.

(18) 186 yds., 11 inches $\div$ 89.

(19) $360^{\circ} \div 365$.

(20) How many copies can be printed off 13 reams, 7 quires, each consisting of 11 sheets?

CHAPTER XI.

PROPERTIES OF NUMBERS, &c.

1. We have hitherto been considering almost exclusively those properties of numbers which belong specially to given numbers; thus, the only number which exceeds 15 by 7 is 22. But some general properties of numbers have been tacitly assumed as true. The expression, "general properties of numbers," requires elucidation. Take, for example, any two numbers, say, 40 and 17; it is evident that $40 + 17 = 17 + 40$; but as this is true of *any* two numbers, we may use more general symbols. Let l and s stand for two different numbers whose values are unknown to us; we still know that $l + s = s + l$, or, in other words, that numbers may be added in any order. This was taken for granted in our rules for addition. We further know that $(l - a) + (s + a) = l + s$, or that the sum of two numbers is not altered by transferring a portion of one of the numbers to the other. This also is accepted as true in "carrying" in addition.

If we are informed that of the two numbers l and s, l stands for the larger of the two, we know that $l - s$ is a possible, and $s - l$ an impossible, quantity. Again, we know that $(l + a) - (s + a) = l - s$, or that "the difference between two numbers will not be altered, if the same quantity be added to both of them." (Ch. IX. Similarly we know and have assumed through Ch. V. and

VI. that $(l + s) \times m = l \times m + s \times m$, or that the sum of two (or more) numbers is multiplied by another number if each of the addenda is so multiplied; thus: $27 \times 5 = 20 \times 5 + 7 \times 5$.

We proceed to other general properties of numbers.

2. If one number can be divided by another number *without remainder*, the divisor is called a MEASURE of the dividend, and the dividend a MULTIPLE of the divisor. Thus 20 can be divided by 5 without remainder, therefore 5 is called a MEASURE of 20, and 20 a MULTIPLE of 5. A length of 20 inches can be *measured* by a rod 5 inches in length, but not by one of 6 inches. 20 can be obtained from 5 by *multiplying* it.

The measures of 6 are 6, 3, 2, 1. It is required to find the measures of 48.

$$\begin{array}{rl} & 48 \\ \hline 1) & (48 \\ 2) & (24 \\ 3) & (16 \\ 4) & (12 \\ 6) & (8 \end{array}$$

Since $1 \times 48 = 48$, 1 and 48 are measures of 48; since $2 \times 24 = 48$, 2 and 24 are measures of 48, and as there are no numbers between 1 and 2, there are no measures between 48 and 24, for any such measure would have to be contained more than once and less than twice. Next take 3; since $3 \times 16 = 48$, 3 and 16 are measures of 48, and, as before, no measure of 48 can lie between 24 and 16, there being no number between 2 and 3. Take 4; since $4 \times 12 = 48$, 4 and 12 are measures of 48. 5 is not a measure of 48. $6 \times 8 = 48$, so that 6 and 8 are measures of 48; and as we have now all measures between 1 and 6, we must have all between 48 and 8; and as 7, the only number between 6 and 8, is not a measure of 48, we have now found *all possible* measures of 48. They are 1, 2, 3, 4, 6, 8, 12, 16, 24, 48.

EXERCISE LVIII.

Find all measures of 10, 12, 16, 20, 36, 60, 100, 112, 120, 144, 240, 360, 960, 1000, 1760, 5760, 7000.

3. *Q.* What is the greatest measure of any number?

A. The number itself.

Q. What is the least?

A. Unity.*

Q. Find the measures of 17.

A. 1 and 17.

A number which has no measures but itself and unity is called a PRIME NUMBER; other numbers are called COMPOSITE NUMBERS.

EXERCISE LIX.

(1) Write out all the prime numbers under 100.

(2) Classify the following numbers into prime and composite numbers: 91, 111, 113, 117, 119, 121, 131, 133.

4. All the multiples of 2 are called EVEN NUMBERS; the others are ODD NUMBERS; and hence the even and the odd numbers lie alternately.

It is evident that 2 is the only prime number that is even; note, however, that though all prime numbers (except 2) are odd, not all odd numbers are prime.

(*a*) Even added to even yields even: thus, $8 + 10 = 18$, for an exact number of twos added to an exact number of twos must be an exact number of twos.

(*b*) Even added to odd yields odd: thus, $8 + 9 = 17$, for an odd number is an exact number of twos $+ 1$, and an exact number of twos added to an exact number of twos $+ 1$ must yield an exact number of twos $+ 1$, i.e. an odd number.

(*c*) Odd added to odd yields even: thus, $7 + 9 = 16$, for an exact number of twos $+ 1$, added to an exact number of twos $+ 1$, must give an exact number of twos $+ 2$, that is, altogether, an exact number of twos. Conversely,

(*d*) Even taken from even leaves even.

(*e*) Even taken from odd leaves odd.

(*f*) Odd taken from even leaves odd.

(*g*) Odd taken from odd leaves even.

* We are only considering whole numbers or integers.

(*h*) Even multiplied by even or odd yields even, for any exact number of twos multiplied by any number must be an exact number of twos.

(*i*) Odd multiplied by odd yields odd, for the multiplier is an even number + 1, so that the multiplicand has to be repeated this even number of times and once more; the result of taking it an even number of times is to give an even number, and adding to this result once the multiplicand, which is odd, the final result will be even + odd, or odd. Conversely,

(*j*) When the dividend is a multiple of the divisor, even divided by odd yields even, because odd must be multiplied by even to yield even.

(*k*) Even divided by even may give either even or odd, because even multiplied by either yields even.

(*l*) Odd divided by odd yields odd.

(*m*) Odd cannot be divisible by even without remainder.

Examination of these rules shews that in Addition and Subtraction likes give even, unlikes odd; and that in Multiplication the product is even unless both factors are odd.

5. If one number is a measure of another number, it will also be a measure of all its multiples; thus 7 is a measure of 14, and is therefore a measure of any number of fourteens. Generally, if a is a measure of b, it is a measure of $m \times b$.

6. If one number is a measure of two other numbers, it will also be a measure of their sum and of their difference. Thus, if there be two numbers, each of which is an exact number of [fives] (say), their sum and their difference must still be an exact number of [fives], i.e. [five] will be a measure both of their sum and of their difference. Generally, if m measures a and b, it will measure $a + b$ and $a - b$. Combining this with what was said in § 5, we see that a number which is a measure of any two numbers is also a measure of the sum or difference of any of their multiples.

7.* If a number measures one *only* of two numbers, it cannot measure either their sum or their difference. If one number is an exact number of [fives], and another is an exact number of [fives] + [three], their sum will be an exact number of [fives] + [three], and their

* §§ 5, 6 and 7 state the general principles of which § 4 gives applications to the measure 2.

difference will be an exact number of [fives] + [three], if the multiple of [five] is the less of the two quantities; or an exact number of [fives] — [three], if the multiple of [five] is the greater of the two quantities, and this difference therefore will not be measured by [five].

If a number measures *neither* of two numbers, it may or may not measure their sum and their difference; [five] will measure the sum of two numbers, if the two remainders together make up another [five]; and their difference, if the remainders are the same in each; e.g., 5 is not a measure of 43 nor of 32, but is a measure of their sum, since 43 = a number of fives + 3, and 32 = a number of fives + 2, so that the sum is an exact number of fives + 5, i.e. is an exact number of fives. Again, 5 is not a measure of 23 nor of 63, but is a measure of their difference, for 63 is a number of fives + 3, and 23 is a number of fives + 3.

8. Divisibility of Numbers.

Examine the number 153576.

(*a*) This is 153576 units or 15357 tens and 6 units; 2 is a measure of 10, and ∴ of 15357 tens (§ 5); but 2 is also a measure of 6; ∴ it is a measure of 15357 tens + 6 units (§ 6), i.e. of 153576. The same reasoning will hold for any number; therefore the divisibility by 2 depends solely upon the figure in the units' place; for every number above ten is a number of tens + some units. The tens are always even, and if the units are also even, we have even + even, which is even (§ 4, *a*); if the units are odd, we have even + odd, which is odd (§ 4, *b*).

Learn by heart: *A number is divisible by 2 if the figure in the units' place is even.*

(*b*) The number 153576 is 1535 hundreds and 76 units. 4 is a measure of 100 (25 × 4 = 100), and ∴ of 1535 hundreds; but 4 is also a measure of 76; ∴ it is a measure of 1535 hundreds + 76 units, i.e. of 153576. If 4 had not measured the quantity after the hundreds, it could not have measured the whole number (§ 7).

Learn by heart: *A number is divisible by* 4 *if the quantity in the units' and tens' places is divisible by* 4.

(*c*) The number 153576 is 153 thousands and 576 units. 8 is a measure of 1000 ($125 \times 8 = 1000$), and ∴ of 153 thousands; but 8 is also a measure of 576, and ∴ of 153 thousands + 576 units, i.e. of 153576. If 8 had not measured the quantity after the thousands, it could not have measured the whole number.

Learn by heart: *A number is divisible by* 8 *if the quantity in the units', tens' and hundreds' places is divisible by* 8.

N.B. 2 is the 1st, 4 the 2nd, and 8 the 3rd power of 2; and in considering the divisibility of numbers by these, we connect 2 with 1 figure, 4 with 2 figures, and 8 with 3 figures, from the right.

(*d*) Reasoning analogous to that given in (*a*) proves that *a number is divisible by* 5 *if the figure in the units' place is either* 5 *or* 0. [Let the pupil write out the proof of this.]

(*e*) $10 = 9 + 1$; $100 = 99 + 1$; $1000 = 999 + 1$, &c.; ∴ every power of 10 is an exact number of nines + 1.

Analyse 153576. It is 6 units, 7 tens, 5 hundreds, 3 thousands, 5 ten-thousands, and 1 hundred-thousand. Breaking up these different values into groups of nines and units, we obtain:

	Nines.	Units.
From the 6 units	—	6
„ 7 tens ($7 \times 9 + 7$)	7 and	7
„ 5 hundreds ($5 \times 99 + 5$)	55 „	5
„ 3 thousands ($3 \times 999 + 3$)	333 „	3
„ 5 ten-thousands ($5 \times 9999 + 5$)	5555 „	5
„ 1 hundred-thousand ($1 \times 99999 + 1$)	11111 „	1

which is an exact number of nines + ($6 + 7 + 5 + 3 + 5 + 1$) units, or an exact number of nines + 27 units; but 27 is itself a multiple of 9; ∴ the whole is divisible by 9. Note that 27 = the sum of the digits in 153576.

Learn by heart: *A number is divisible by* 9 *if the sum of its digits* is divisible by* 9.

(*f*) Reasoning analogous to that just given shews that *any number is an exact number of nines + the sum of its digits.* The nines must be divisible by 3; if, therefore, the sum of the digits is also divisible by 3, the whole is divisible by 3.

* The word digit is derived from the Latin *digitus*, a finger, the fingers being the natural counters.

Learn by heart: *A number is divisible by* 3 *if the sum of its digits is divisible by* 3.

NOTE. Every number divisible by 9 is divisible by 3; but not every number divisible by 3 is divisible by 9.

(*g*) Any number which is divisible by 6 will stand both the tests of 2 and 3; for sixes can be broken up into twos or into threes. Conversely, only numbers divisible by 6 can stand both these tests, for a number not divisible by 6 must be some sixes + 1, 2, 3, 4 or 5, no one of which remainders can be arranged both in twos and threes.

Learn by heart: *A number is divisible by* 6 *if it is divisible both by* 2 *and by* 3.

(*h*) $10^2 = 100 = 99 + 1 = 9 \times 11 + 1.$
$10^4 = 10000 = 9999 + 1 = 909 \times 11 + 1.$
$10^6 = 1000000 = 999999 + 1 = 90909 \times 11 + 1$, &c.,

or any even power of 10 diminished by 1 is a multiple of 11.

Take the number 1493756, which is 56 units + 37 hundreds + 49 ten-thousands + 1 million. Breaking up these different values into groups of elevens and units, we obtain:

		Elevens.		Units.
56 units	=	—		56
37 hundreds	=	37 × 9	and	37
49 ten-thousands	=	49 × 909	„	49
1 million	=	90909	„	1
				143

which is some elevens + 143 units; but 143 is itself a multiple of 11, ∴ 11 is a measure of the whole number 1493756. Hence if the sum of the successive pairs of digits be divisible by 11, the number is divisible by 11. Instead of writing the digits in pairs, it is shorter to add them up alternately, considering the digits in the odd (1st, 3rd, 5th, &c.) places as units, and those in the even places as tens; e.g. in 1493756:

Wording: 6, 13, 22, 23′, carry 2; 7, 10, 14′. *Ans.* 143.

Learn by heart: *A number is divisible by* 11 *if the sum of its digits, added up alternately, beginning at the units' place, is divisible by* 11.

(*i*) Reasoning analogous to that in (*g*) shews that a number is divisible by 12 if it will stand the tests of 3 and of 4. [Let the pupil write out the proof of this.]

[(*j*) For divisibility by 7, many tests have been suggested, but no one of them is shorter than actual trial. The pupil may exercise his ingenuity in proving the following methods:

(α) Break the given number into pairs as for 11; divide the pair of highest designation by 7, double the remainder, if any, and add this product to the next pair; divide this sum by 7, and again add the double of its remainder to the next pair, and so on; if the last remainder is 0 or a multiple of 7, the whole is so.

(β) Break into pairs as before; to the pair of lowest designation add twice the next, four times the third, once the fourth, twice the fifth, four times the sixth, &c.; if the result is divisible by 7, the number is so.

(γ) If the difference between double the units and once the tens is 0, or a multiple of 7, the number itself is a multiple of 7. For short numbers this last method is useful.]

Exercise LX.

Determine by inspection the measures under 13 of the following numbers: 504, 405, 315, 168, 451, 512, 98, 1080, 9999, 864, 1296, 6144, 7020, 7040, 33264, 142857, 999999, 2520.

9. Casting out Nines.

We can now demonstrate the truth of the rules given above for testing the accuracy of the results in Addition, Subtraction, Multiplication and Division.

It was proved in § 8 (*e*, *f*) that "any number is an exact number of nines + the sum of its digits." Hence the remainder of a number divided by 9 may be found by casting out the nines from the sum of the digits. In the method given in Ch. II. § 4, the nines were rejected as fast as they were obtained, which is obviously the same as casting them out at the end.

Addition :

655	7
1362	3
34052	5
15	6
36084	3′

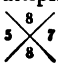
The effect of casting out nines must evidently be the same, whether we cast them out from the addenda separately or from their sum.

Subtraction.

a.	4371	6	*b.*	43726	4	*c.*	56143	1
	456	6		18237	3		12345	6
	3915	0′		25489	1′		43798	4′

In (*a*) we take a certain number of nines + 6 from another number of nines + 6, and must therefore have an exact number of nines left. In (*b*) we take a certain number of nines + 3 from another number of nines + 4, and must therefore have an exact number of nines + 1 left. In (*c*) we take a certain number of nines + 6 from another number of nines + 1 or (as it may be called) a number of nines + 10, and we must therefore have a certain number of nines + 4 left.

Multiplication :



a. 37598×7

263186

b. 37598×457

```
  263186
 187990
150392
--------
17182286
```

In (*a*) the multiplicand is a certain number of nines + 5, which, multiplied by 7, is a certain number of nines + 7 × 5 or 35. In (*b*) the same multiplicand is to be multiplied by a certain number of nines + 7; multiplication by nines yields nines, and the multiplication by 7 gives, as in (*a*), a certain number of nines + 5 × 7 or 35, and this is in each case a certain number of nines + 8, ∴ the product should be a multiple of nine + 8.

Division :

a. 3586)168542(47
25102
....

b. 3586)168665(47
25225
123

In (*a*) the divisor is contained 47 times exactly in the dividend; therefore the dividend = 47 × the divisor, and the test for multiplication applies. In (*b*) the divisor 3586 taken 47 times from the dividend leaves the remainder 123; therefore 47 × 3586 + 123 = the dividend 168665, whence the truth of the rule is apparent.

10. Casting out elevens may be used with advantage as an additional test of accuracy. Elevens may be cast out in two ways, either by alternate addition, or by successive subtractions from left to right.

It follows from the reasoning of (*h*) page 134, that every number is a multiple of eleven, + the sum of its digits added up alternately, beginning at the units' place. If, then, this addition is made, and if necessary again made on the result, till at last a number is obtained consisting of not more than two digits, elevens have so far been cast out, and the final remainder can be obtained by simple division by eleven. Take, for example, 6834567.

Wording: 7, 12, 15, 21′, carry 2; 8, 12, 2′0′; 201; again, 1, 3′, remainder 3. If, then, 6834567 be divided by 11, the remainder will be 3.

Take 683456797.

Wording: 7, 14, 19, 22, 28′, carry 2; 11, 17, 21, 2′9′; 298; again, 8, 10′, carry 1; 1′0′; 100; again, 0, 1′; remainder, 1, i.e. 683456797 ÷ 11, leaves remainder 1.

Take 68345679745.

Wording: 5, 12, 19, 24, 27, 33′, carry 3; 7, 16, 22, 26, 3′4′; 343; again, 3, 6′; 4′; 46; 46 ÷ 11 leaves 2; i.e. 68345679745 ÷ 11 leaves remainder 2.

This division (disregarding the quotient) may be performed by simple subtraction thus: 11 into 68, 6, carry 2 (this 2 being 8 − 6); into 23, 2, carry 1 (1 being 3 − 2), and so on.

Wording: 6 from 8, 2; from 3, 1; from 4, 3; from 5, 2; from 6, 4; from 7, 3; from 9, 6; from 7, 1; from 4, 3; from 5, 2′; or shorter, 2, 1, 3, 2, 4, 3, 6, 1, 3, 2′.

Take 475615165. 1st, by alternate addition:

Wording: 5, 6, 7, 12, 16′, carry 1; 7, 12, 18, 2′5′; 256; again, 6, 8′, 5′; 58; 58 ÷ 11 yields remainder 3; or shorter, 5 from 8 = 3′.

By successive subtractions :

Wording: 4 from 7, 3; from 5, 2; from 6, 4; 4 from 1? here is a difficulty, which can be got over thus: subtract backwards, 1 from 4, 3, and *add* this number to the figure after 1, viz. 5; $3+5=8$; because the three numbers 415 are four hundred and fifteen units of a certain kind; if from one hundred units, elevens be cast out, there is a remainder 1, and from four hundred units the remainder is 4; this 4 added to the 15 makes 19, which leaves 8, $(9-1)$. This subtraction of 1 may be done before the addition: 8 from 1?—1 from $8=7$; $7+6=13$; rejecting 11 from 13, leaves 2; 2 from $5=3'$; or shorter, 3, 2, 4, $3+5=8$, $7+6=13$, 2, $3'$, as above, $\therefore$ $4756151 65 \div 11$ leaves remainder 3.

Take 6382417412. By alternate addition :

Wording: 2, 6, 7, 9, $12'$, carry 1; 2, 9, 13, 21, $2'7'$; 272; 2, $4'$, $7'$; 74; $74 \div 11$ leaves 8, or 7 from 4 cannot be done; add 11 to 4, 7 from $11+4=8$.

By subtractions :

Wording: 3 from 6, 3; $3+8=11$, 0; 2 from 4, 2; 1 from 2, 1; $1+7=8$; 4 from 8, 4; $4+1=5$; 5 from 2?—5 from $2+11=8'$, as above; or shorter, 3, 11, 0, 2, 1, 8, 4, 5, $8'$.

11. The whole subject of criteria for divisibility of numbers may sometimes with advantage be treated from the wider point of view of ascertaining the remainders of the division without the trouble of finding the quotient. In the case where the remainder is 0, the number is divisible by the divisor: thus

a.	Every number is a	multiple of	$2+$ the quantity in the units' place.
b.	,,	,,	$3+$ the sum of its digits.
c.	,,	,,	$4+$ the quantity in the units' and tens' places.
d.	,,	,,	$5+$ the quantity in the units' place.
e.	,,	,,	$8+$ the quantity in the units', tens' and hundreds' places.
f.	,,	,,	$9+$ the sum of its digits.
g.	,,	,,	$10+$ the quantity in the units' place.
h.	,,	,,	$11+$ the sum of its digits added alternately, beginning at the units' place.

If, then, in this table the quantities following the sign + be alone regarded, multiples of the numbers to the left of the sign will have been cast out: *e.g.* cast out fours from 1257639. $1257639 =$ a multiple of $4+39$, and $39 \div 4$ leaves remainder 3.

Accordingly casting out twos, fives, tens, fours and eights, would *add to the probability* of correctness only in the last one, two or

three figures as the case may be, and for this reason it is best to choose nines and elevens, as the criteria have regard to all the digits.

In cases of importance, especially where much subsequent work depends upon the accuracy of a particular result, it is advisable to apply both the tests to this result.

[12. Divisibility of Numbers in other simple Scales of Notation. There are three obvious cases to be considered.

Case I. Where the proposed divisor is a measure of any power of the radix. If it measures the first power of the radix, the criterion of divisibility is the units' figure; if it measures the units' figure, it will measure the given number, and if not, not. (Cf. § 8, *a*, *d*.) If it measures the second power, the criterion is the last two figures; if the third power, the last three. (Cf. § 8, *b*, *c*.)

Case II. Where the proposed divisor is 1 less than the radix, or a factor of this number. The criterion is the sum of the digits. (Cf. § 8, *e*, *f*.)

Case III. Where the proposed divisor is 1 more than the radix. The criterion is the sum of the digits added up alternately, beginning at the units' place (§ 8, *h*), or the result of successive subtractions, as in casting out elevens in the decimal scale (§ 10).

Exercise LXI.

By what numbers not exceeding the radix are the following divisible?

(1) 23054, 12304, 5523 (senary).

(2) 112235, 16245, 43700 (octonary).

(3) 444444, 52836, 78780 (nonary).

(4) 7347*t*2, *tteet*, 486310 (duodecimal).]

13. Resolution into Prime Factors. Every number is either prime or is the product of two or more prime numbers or their powers. Thus 5 is prime; 6 is the product of 2 and 3, which are prime; 8 is $2 \times 2 \times 2 =$ the third power of 2; and $12 = 3 \times 2 \times 2 = 3$ times the second power of 2.

Find the prime factors of 23760. In § 8 we have given rules for determining by inspection the presence of the prime factors, 2, 3, 5, 7, 11. Applying these to the number 23760, we find that it is divisible by 2 four times successively, by 3 three times successively,

by 5 and by 11; therefore $23760 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 11$. It is convenient to write this, $2^4 \times 3^3 \times 5 \times 11$.

Mod. op.:

$$\begin{array}{r} 2\underline{)23760} \\ 2\underline{)11880} \\ 2\underline{)5940} \\ 2\underline{)2970} \\ 3\underline{)1485} \\ 3\underline{)495} \\ 3\underline{)165} \\ 5\underline{)55} \\ 11 \end{array}$$

Ans. $2^4 \times 3^3 \times 5 \times 11$.*

EXERCISE LXII.

Resolve into prime factors:

(1) 6	(5) 36	(9) 120	(13) 1320	(17) 7000
(2) 8	(6) 40	(10) 143	(14) 1760	(18) 8140
(3) 15	(7) 91	(11) 240	(15) 1845	(19) 8712
(4) 16	(8) 96	(12) 720	(16) 5760	(20) 1848

When all the prime factors of a number are found, all its other factors can be determined from them, since these are only products of the former. For example, $84 = 2^2 \times 3 \times 7 = 2 \times 2 \times 3 \times 7$; the only other factors of 84 are those which can be compounded from these prime factors, viz.

$2 \times 2 = 4$	$2 \times 7 = 14$	$2 \times 2 \times 3 = 12$	$2 \times 3 \times 7 = 42$
$2 \times 3 = 6$	$3 \times 7 = 21$	$2 \times 2 \times 7 = 28$	$2 \times 2 \times 3 \times 7 = 84$

Hence the factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84; as we might have found in a more convenient manner, as in § 2.

In Exercise LXII., such numbers only have been chosen as have not more than one prime factor exceeding 11, and none exceeding 100; but if a number be proposed not subject to these limitations, how are we to determine its prime factors?

Find the prime factors of 867. The factor 3 is obvious: $867 = 3 \times 289$.

$$\begin{array}{r} 3\underline{)867} \\ 289 \end{array}$$

289 is not divisible by 2, 3, 5, 7 or 11, but we are not yet justi-

* *It is assumed here* that a number can be resolved into prime factors in only *one way. It is, however, proved* in the foot-note to pp. 151, 152.

fied in declaring it prime, as it may have factors exceeding 11. Try 13, and we find the remainder 3.

$$\begin{array}{l} 13)289(22 \\ \quad\ \ 29 \\ \quad\ \ \ 3 \end{array}$$

The next prime number is 17;* on trying which, we find that it is contained exactly 17 times, $\therefore$ $867 = 3 \times 17^2$.

$$\begin{array}{l} 17)289(17 \\ \quad\ 119 \end{array}$$

Since $289 = 17 \times 17$, 289 can have no factor higher than 17. since such a factor would have to be contained in 289 less than 17 times, and 289 would therefore be divisible by some number less than 17, which we have found not to be the case.

Find the factors of 283. 283 is not divisible by 2, 3, 5, 7 or 11. Try 13. There is a remainder 10.

$$\begin{array}{l} 13)283(21 \\ \quad\ \ 23 \\ \quad\ \ 10 \end{array} \qquad 21 \times 13 = 273,\ 22 \times 13 = 286$$

Try 17. There is a remainder 11.

$$\begin{array}{l} 17)283(16 \\ \quad\ 113 \\ \quad\ \ 11 \end{array} \qquad 16 \times 17 = 272,\ 17 \times 17 = 289$$

Therefore 283 can have no factor exceeding 17, for such a factor would have to be contained less than 17 times, that is to say, 283 would be divisible by some number less than 17, which we have found not to be the case. Therefore 283 is prime. Hence, in seeking the factors, prime or composite, of any number, we need only try prime numbers until the quotient is equal to or less than the divisor. Thus, for numbers less than 100, we need only try primes under 10.

EXERCISE LXIII.

Classify the following numbers into prime and composite, and resolve each composite number into its prime factors: 101, 765, 169, 247, 2109, 365, 1367, 1867, 4019, 3059, 483, 99, 999, 9999, 99999, 999999.

* No composite number need be tried, it being "compounded" of earlier prime numbers, which have already been tried.

14. Greatest Common Measure.

The measures of 20 and 30 respectively are: 1, 2, 4, 5, 10, 20; 1, 2, 3, 5, 6, 10, 15, 30. We see that these two numbers have the measures 1, 2, 5, 10, in common, while 4 and 20 belong only to 20; 3, 6, 15 and 30 only to 30.

Learn by heart: *The measures that two or more numbers have in common are called their* Common Measures, *and the greatest of these is called their* Greatest Common Measure, *which is indicated by the letters* G. C. M.

15. The only common measure of 8 and 15 is 1, and these two numbers are therefore said to be "prime to each other," although neither is a "prime number."

Learn by heart: *Numbers whose only common measure is unity are said to be* PRIME TO EACH OTHER, *even though they be not prime numbers.*

16. Find G. C. M. of 108 and 1440.

First method. Find *all* the factors of the smaller number.

1) (108
2) (54
3) (36
4) (27
6) (18
9) (12

The largest of these which also measures 1440 is of course the G.C.M. Beginning our trial divisions of 1440 with the greatest of these factors, viz. 108 itself, we find that it is not a measure of 1440.

```
108)1440(13
     360
      36
```

Next try 54; this also is not a measure.

```
54)1440(26
    360
     36
```

Now try 36, which is contained exactly 40 times, and is therefore the G.C.M. required.

```
36)1440(40
     ••
```

EXERCISE LXIV.

Find G.C.M. of:

(1) 84 and 96	(5) 120 and 150	(9) 28 and 42
(2) 48 and 144	(6) 38 and 57	(10) 66 and 99
(3) 32 and 60	(7) 28 and 49	(11) 100 and 175
(4) 45 and 28	(8) 141 and 74	(12) 180 and 240

This method, which is so obvious, is convenient for small numbers, but ceases to be so when the factors cannot be readily detected.

Second method. Resolve both numbers into their prime factors.

$$\begin{array}{r} 2\underline{)108} \\ 2\underline{)54} \\ 3\underline{)27} \\ 3\underline{)9} \\ 3 \end{array} \qquad \begin{array}{r} 2\underline{)1440} \\ 2\underline{)720} \\ 2\underline{)360} \\ 2\underline{)180} \\ 2\underline{)90} \\ 3\underline{)45} \\ 3\underline{)15} \\ 5 \end{array}$$

$$108 = \dot{2} \times \dot{2} \times \dot{3} \times \dot{3} \times 3$$
$$1440 = \dot{2} \times \dot{2} \times 2 \times 2 \times 2 \times \dot{3} \times \dot{3} \times 5$$

From § 13 it is evident that the product of all the prime factors which they have in common is the G.C.M. required. These are $2 \times 2 \times 3 \times 3 = 36$.

If 108 and 1440 be divided by their G.C.M. 36, the quotients 3 and 40 ($2 \times 2 \times 2 \times 5$) are prime to each other, as they consist of those factors which are not common to both numbers; and generally,

If two numbers are divided by their G.C.M., the quotients are prime to each other, and it often affords a convenient test for G.C.M. to examine whether the quotients are prime to each other.

Again: as the G.C.M. is made up of all the common factors of the two numbers, every measure of the G.C.M. is a common measure of the two numbers, and they have no other measure in common.

EXERCISE LXV.

Find G.C.M. of

(1) 512 and 240	(5) 112 and 28	(9) 2100 and 2240
(2) 1760 and 990	(6) 77 and 231	(10) 210 and 1008
(3) 5760 and 7000	(7) 840 and 1440	(11) 1485 and 2160
(4) 212 and 504	(8) 360 and 900	(12) 7040 and 7392

Third method. Find G.C.M. of 4063 and 4541. In this case, the above methods would be too laborious, as we cannot determine the factors by mere inspection.

Although as yet the G.C.M. is unknown, the following facts concerning it are known:

(*a*) It cannot possibly exceed the smaller of the given numbers (§ 3). Hence,

(*b*) If the smaller is contained in the greater, it is the G.C.M. of the two numbers.

4063) 4541 (1
478

On trial, we find the remainder 478.

(*c*) Any common measure of the two numbers must measure the difference of any of their multiples (§ 6), and hence must measure 478, therefore the G.C.M. we are seeking must be *a* common measure of 478 and 4063. Let us, therefore, find a common measure of the numbers 478 and 4063. Reasoning as before, the G.C.M. we are seeking must measure the difference of any of the multiples of 478 and 4063. Take from 4063 the largest possible multiple of 478, i.e. divide 4063 by 478. Remainder 239.

478) 4063 (8
239

Hence the G.C.M. required must measure 239 and 478. As before, divide 478 by 239. We find $478 = 2 \times 239$.

239) 478 (2
. . .

We have now shewn that *every* measure of 4063 and 4541 must also measure 239; we proceed to prove the converse, viz., that every measure of 239 must also measure 4063 and 4541. Every measure of 239 measures also its multiple 478 (§ 5), and therefore also $8 \times 478 + 239$ (§ 6), i.e. 4063; and therefore also $1 \times 4063 + 478$ (§ 6), i.e. 4541. But the greatest measure of 239 is 239, which is therefore the G.C.M. required. The sum is commonly worked in the following form:

```
4063) 4541 (1
      4063
       478) 4063 (8
            3824
             239) 478 (2
                  478
                  ...
```

By the method of Ch. IX. § 13, the sum would look thus:

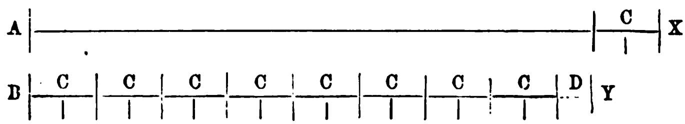
```
4063) 4541 (1
       478) 4063 (8
             239) 478 (2
                  ...
```

This can be still further contracted by dividing alternately from right to left and left to right, writing the quotients in order at the top of the T.

1, 8, 2

4063	4541
239	478
	...

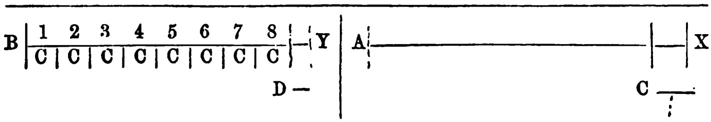
The process here followed admits of the following illustration:



It is required to find the greatest length which is contained an exact number of times in each of the straight lines AX and BY. Since this required length (which we may call L) is to measure both AX and BY, it must measure their difference, viz. C ——, and as L measures C and BY, it must measure any number of times C and the difference between BY and this multiple of C; take from BY as many times C as possible, and suppose there is a remainder, D —, after taking it eight times. As before, L must measure both C and D, and suppose we find that D is contained exactly (say twice) in C, D will be the length required; for since D measures C, it measures $8 \times C + D$ or BY; and since it measures BY, it also measures $BY + C$ or AX, and is therefore a common measure of AX and BY; and it has been shewn that L must

measure D; and as D is the greatest measure of itself, D is the G.C.M. required.

1, 8, 2



EXERCISE LXVI.

Find G.C.M. of:

(1) 2536 and 3487
(2) 2479 and 3589
(3) 3045 and 6195
(4) 8823 and 11937
(5) 568 and 712
(6) 419 and 52301
(7) 11023 and 6493
(8) 27671 and 408870
(9) 35143 and 10283
(10) 232353 and 39699
(11) 78473 and 94653
(12) 2760 and 4485
(13) 1177 and 2675
(14) 14141 and 16289
(15) 85359 and 86128
(16) 44323 and 61087
(17) 17596 and 26145
(18) 1485 and 2160
(19) 7040 and 7392
(20) 999999 and 571428

Fourth method. On examining the process last given, we notice that the question of finding G.C.M. of two large numbers is reduced to that of finding two smaller numbers having the same G.C.M. If, therefore, one of the two given numbers has a prime factor not contained in the other, it may be thrown out at once. Again, a factor, prime or not, which is common to both, may be taken out and set aside for ultimate multiplication into the last divisor; and all this may be done at any stage of the process. This method will be found useful where the factors can be detected by inspection.

Find G.C.M. of 976800 and 9990.

10) 976800	9990	1st stage
3) 97680	999	2nd ,,
3256	333	3rd ,,
407	37	4th ,,
37		

$\therefore$ G.C.M. $= 37 \times 3 \times 10 = 1110$.

1st stage. Take out and preserve the common factor 10.

2nd stage. Reject $5 \times 2 = 10$ from the left, neither factor being common; take out and preserve the common factor 3.

3rd stage. Reject $2 \times 2 \times 2 = 8$ from the left, and $3 \times 3 = 9$ from the right, as not common.

4th stage. Reject 11 from the left.

Find G.C.M. of 250387 and 41041.

6, 9, 1	
250387	41041
41) 4141	
101	1001
9	92

G.C.M. = 41.

In this example, we commenced as usual, but the first division revealed the common factor 41, the two quotients 101 and 1001 proving to be prime to each other. 41 is G.C.M. required.

Exercise LXVII.

Find G.C.M. of:

(1) 1679 and 1932	(9) 2268 and 3348
(2) 1003 and 2419	(10) 1189 and 2146
(3) 33853 and 35017	(11) 94653 and 78473
(4) 533 and 1189	(12) 2993 and 3869
(5) 33787 and 34691	(13) 768 and 16777216
(6) 11009 and 12827	(14) 5115 and 7254
(7) 4189 and 4307	(15) 324 and 456
(8) 4489 and 5293	(16) 269178 and 352002

16. If the G.C.M. of more than two numbers be required, we must first find that of any two of them; then the G.C.M. of this result and a third number; then of this second result and a fourth, and so on.

Find G.C.M. of 4994, 7491, 9988, 12485, 16571.

						73	
11) 4994	7491	11) 227 × 11	9998	11) 11 × 227	12485	11 × 227	16571
454	681	227	908	227	1135	227	681
227	227		227		227		...
G.C.M. 227 × 11		G.C.M. 11 × 227		G.C.M. 11 × 227		G.C.M. 227	
						Ans. 227.	

227 is therefore the G.C.M. of all the numbers.

Find G.C.M. of 7326, 8547, 9768, 22755.

	7
3) 7326	8547
11) 2442	2849
222	259
111	...
37	

G.C.M. $3 \times 11 \times 37$

	8
3) $3 \times 11 \times 37$	9768
11) 11×37	3256
37	296
	...

G.C.M. $3 \times 11 \times 37$

	41
3) $3 \times 11 \times 37$	22755
11×37	7585
37	1517
	37
	..

G.C.M. $3 \times 37 = 111$

Ans. 111.

NOTE.—If of a series of numbers there be one which measures each of the others, it is the G.C.M. of the series.

EXERCISE LXVIII.

(1) Find G.C.M. of 12, 24, 36.

(2) „ 2255, 4305, 6355, 9020, 10455.

(3) „ 68, 17, 102, 34.

(4) „ 909, 1414, 2323, 4242, 2121.

(5) „ 1521, 585, 4095, 3393, 10764, 4563.

(6) „ 132288, 107328, 138216, 97344.

(7) „ 740, 333, 296.

(8) Find the largest number of which the following are multiples: 833, 1785, 1309.

(9) An exact number of shares, all at the same price, was bought with each of the following sums: £87. 6*s.* 3*d.*, £134. 18*s.* 9*d.*, £341. 6*s.* 3*d.* Find the highest possible price of each share.

(10) Two distances of 901 and 1037 miles respectively are portioned off into equal daily journeys. Find the smallest number of days in which the journeys can be accomplished.

(11) A court 6 yds., 2 ft., 7 in. long, and 5 yds., 2 ft., 5 in. broad, is to be paved with *square* tiles. Find the largest possible size of the tiles, and how many are required.

17. LEAST COMMON MULTIPLE.

The multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, 27, &c.; the multiples of 4 are 4, 8, 12, 16, 20, 24, 28, &c. We see that

these two numbers have the multiples 12, 24, 36, &c., in common, while 3, 6, 9, 15, 18, &c., belong only to 3, and 4, 8, 16, 20, &c., only to 4.

Learn by heart: *The multiples that two or more numbers have in common are called their common multiples, and the least of these is called their* LEAST COMMON MULTIPLE, *which is indicated by the letters* L. C. M.

18. Find L.C.M. of 6 and 7.

Write out simultaneously the multiples of 6 and 7:

6, 12, 18, 24, 30, 36, 42;
7, 14, 21, 28, 35, 42;

until we find a number (42) in one series which has already occurred in the other. This number 42 is the L.C.M. required, which means not only that $7 \times 6 = 6 \times 7$, but that seven is the least number of sixes which is also a number of sevens. Other common multiples will be found further on in the series, namely 2×42, 3×42, and the other multiples of 42. No number between once 42 and twice 42 can be a common multiple of 6 and 7; for if 6 and 7 measured some number between 42 and 84, they would also measure the difference between that number and 42, which must be less than 42; but we have just seen that 42 is the *least* common multiple of 6 and 7. The same mode of reasoning shews that no common multiple of 6 and 7 lies between twice 42 and three times 42; and, generally, any common multiple of two numbers is a multiple of their L. C. M.

EXERCISE LXIX.

Find L. C. M. of

(1) 5 and 6	(3) 6 and 11	(5) 9 and 10
(2) 5 and 9	(4) 7 and 3	(6) 2 and 3

The *smallest* (integral) common measure of integers is 1. There cannot be a *greatest* common multiple.

Two numbers must have *a* common multiple; for 2×3 or 6, must be a multiple of 2 and of 3; similarly, 17×19 or 323, must contain 17, 19 times, and 19, 17 times; and the question arises whether this product of two numbers is always not only *a* common multiple, but their *least* common multiple.

Examine the following series, consisting of the several prime numbers and their multiples:

2																
	3															
4																
		5														
6	6															
			7													
8																
	9															
10		10														
				11												
12	12															
					13											
14			14													
	15	15														
16																
						17										
18	18															
							19									
20		20														
	21		21													
22				22												
								23								
24	24															
		25														
26					26											
	27															
28			28													
									29							
30	30	30														
										31						
32																
	33			33												
34						34										
		35	35													
36	36															
											37					
38							38									
	39				39											
40		40														
												41				
42	42		42													
													43			
44				44												
	45	45														
46								46								
														47		
48	48															
			49													
50		50														
	51					51										
52					52											
															53	
54	54															
		55		55												
56			56													
	57						57									
58									58							
																59
60	60	60														

These series evidently contain every number up to 60; if prime *at the head of a column*, if composite lower down. Those com-

posite numbers which are powers of prime numbers, will appear only in the column of that prime number, the others will appear in two or more columns; thus 8, which is $2 \times 2 \times 2 = 2^3$, appears only under two, while 6 appears under 2 and 3, and 42 under 2, 3, 7. If the columns were carried far enough in both directions, they could be made to include any number whatsoever. All lists of multiples of composite numbers are dispersed at regular intervals through these series; thus, 6 and its multiples are the 3rd, 6th, 9th, &c., numbers of the (2) column, and the 2nd, 4th, 6th, of the (3) column. 8 and its multiples are only found in the (2) column in the 4th, 8th, 12th, &c., places; 42 and its multiples would be found dispersed with similar regularity in the (2), (3) and (7) columns.

The L.C.M. of 2 and 3 is 6, of 3 and 5 is 15, of 3 and 7 is 21, of 2 and 17 is 34, and so on.

Learn by heart: *The* L.C.M. *of two prime numbers is their product.*

The L.C.M. of 8 and 3 is also their product 24; 8 being only in the (2) and 3 only in the (3) column, the earliest point at which they can coincide is $3 \times 8 = 24$. Similarly, the L.C.M. of 8 and 9 will be found to be 72. The L.C.M. of 6 and 5 or of 2×3 and 5 is 30, for though 6 appears in two columns, 5 does not appear in either of them until we reach 6×5 or 30. Similarly, the L.C.M. of 15 and 4, or of 3×5 and 2×2, is 15×4, or 60. Again, the L.C.M. of 15 and 14, or 3×5 and 2×7, is $15 \times 14 = 210$. We find, then, that if the two given numbers have no factor in common, i.e. if they are *prime to each other*, the L.C.M. is their product. We may therefore extend the last rule.

Learn by heart: *The* L.C.M. *of two numbers prime to each other is their product.**

The L.C.M. of 8 and 16 is 16, for it is a multiple of 8, and no less number would be a multiple of 16.

* This rule is axiomatic in its character, and is accordingly often assumed without proof. We have endeavoured by tabular exhibition to bring it home more closely to the mind of the beginner. It admits, however, of the following proof:

(1) It is proved that if m measures a and b, it measures their G.C.M. (P. 143.)

(2) If m measures $a \times b$ and is prime to a, it must measure b, for it measures $a \times b$ (hyp) and evidently also $m \times b$, and $\therefore$ the G.C.M. of $a \times b$ and $m \times b$, but this G.C.M. is b, for if a greater number, say c, were this G.C.M., b would measure

Learn by heart: *If of two numbers one is a multiple of the other, it is the* L.C.M. *of the two.*

Find L.C.M. of 6 and 15. $6 = 2 \times 3$, $15 = 5 \times 3$, $\therefore$ the multiples of each of the given numbers will be found in the (3) column, those of 6 occupying the 2nd, 4th, 6th, 8th, 10th, &c., places, and those of 15 the 5th, 10th, 15th, &c., places. Hence the first place where the multiples coincide is the 10th place, i.e. $10 \times 3 = 30$, because 2 and 5 being prime to one another, 10 is their L.C.M.

Find L.C.M. of 30 and 12. $30 = 2 \times 3 \times 5$, $12 = 2 \times 2 \times 3$. Hence the multiples of 30 and 12 are both in the (2) and in the (3) columns, therefore they would all be in a (6) column, if such were made, or selected from the columns before us. The multiples of 30 would occupy the 5th, 10th, &c., places of such a column, those of 12 the 2nd, 4th, 6th, &c., places; $\therefore$ (as in the case of 6 and 15) $6 \times 5 \times 2$ is L.C.M. of 30 and 12. We have fixed upon the common factor 6 as the largest among whose multiples 30 and 12 could both be found, i.e. the G.C.M. of 30 and 12. Hence to find L.C.M. of two numbers, divide them both by their G.C.M., and the continued product of this G.C.M. and the two quotients will be the L.C.M. required.

Find L.C.M. of 85 and 187.

85	187
17	17

17) 85 (5
      ..

17) 187 (11
      17
      ..

$\therefore 5 \times 11 \times 17 = 935$ is L.C.M. required.

One of these divisions might have been dispensed with, since the step of dividing by 17 has to be retraced by subsequent multiplication. We might therefore have proceeded thus:

17	85, 187
	5
	5×187

or

17	85, 187
	11
	11×85

i.e. $5 \times 187 = 11 \times 85 = 935$. *Ans.* 935 is L.C.M. required.

c; say $c = k \times b$, $\therefore a$ and m would be divisible by k, and would not be prime to each other as they were supposed to be.

(3) Since m measures b, the least value of b is m, and $\therefore a \times m$ is the L.C.M. of a and m.

(4) From (2) it follows that a number can be resolved into prime factors in only one way. Let $N = a \times b \times c \times d$, where a, b, c, d, are primes, and let p be another prime number, then N is not divisible by p; for if it were so, p, being prime to a, must measure $b \times c \times d$; then being prime to b, it must measure $c \times d$; and being prime to c, it must measure d, to which by hypothesis it is prime.

Learn by heart: ***To find* L. C. M. *of two numbers, divide either number by their* G. C. M., *and multiply this quotient by the other number.***

Find L. C. M. of:

EXERCISE LXX.

(1)	60 and 90	(11)	345 and 346
(2)	75 and 100	(12)	960 and 1000
(3)	80 and 105	(13)	180 and 150
(4)	25 and 75	(14)	801 and 890
(5)	25 and 21	(15)	555 and 370
(6)	13 and 12	(16)	120 and 320
(7)	365 and 657	(17)	424 and 583
(8)	5000 and 6000	(18)	319 and 407
(9)	25 and 30	(19)	1679 and 1932
(10)	345 and 690	(20)	1003 and 2419

19. To find L. C. M. of more than two numbers, we may first find that of any pair, then L. C. M of the result and a third number, and so on; but in most cases this method would be found unnecessarily cumbrous.

20. Find L. C. M. of 8, 24, 48, 96. Since 96 is a multiple of each of the other numbers, and is the least multiple of itself, it is L. C. M. of the series. Hence, if of a series of numbers there is one a multiple of each of the others, this one is the L. C. M. of the series.

21. Examine the number 840. Resolve it into its prime factors.

$$\begin{array}{r|l} 2 & 840 \\ \hline 2 & 420 \\ \hline 2 & 210 \\ \hline 3 & 105 \\ \hline 5 & 35 \\ \hline & 7 \end{array} \qquad 840 = 2 \times 2 \times 2 \times 3 \times 5 \times 7$$

If 840 is divided by any of these factors, or by the product of two or more of them, the remaining factors will give the quotient, thus:

$840 \div 2 = 2 \times 2 \times 3 \times 5 \times 7 = 420$. $840 \div 3 \times 7 = 2 \times 2 \times 2 \times 5 = 40$, and so on,

from which we see that 840 is a multiple of any number whose factors are selected from the series 2, 2, 2, 3, 5, 7.

Again, 840 is *not* a multiple of a number whose factors are not *all* to be found in this series.

Find L. C. M. of 30, 56, 42, 140. Resolve each of these numbers into its prime factors.

2)30	2)56	2)42	2)140
3)15	2)28	3)21	2)70
5	2)14	7	5)35
	7		7

$$30=2\times3\times5 \qquad 42=2\times3\times7$$
$$56=2\times2\times2\times7=2^3\times7 \qquad 140=2\times2\times5\times7=2^2\times5\times7$$

These prime factors can all be selected from the series, 2, 2, 2, 3, 5, 7; therefore, from what we have just said, $2 \times 2 \times 2 \times 3 \times 5 \times 7$, or 840, is *a* common multiple of them all, and it is the *least* common multiple, because no one of these factors can be dispensed with.

Find L. C. M. of 224, 180, 910, 1404, 2025.

2)224	2)180	2)910	2)1404	3)2025
2)112	2)90	5)455	2)702	3)675
2)56	3)45	7)91	3)351	3)225
2)28	3)15	13	3)117	3)75
2)14	5		3)39	5)25
7			13	5

$$224=2^5\times7 \qquad 1404=2^2\times3^3\times13$$
$$180=2^2\times3^2\times5 \qquad 2025=3^4\times5^2$$
$$910=2\times5\times7\times13$$

From these select the highest power of each prime number, viz. 2^5 (from 224), 3^4 (from 2025), 5^2 (from 2025), 7 (from 224 or 910), 13 (from 910 or 1404).

$$2^5\times3^4\times5^2\times7\times13=2\times2\times2\times2\times2\times3\times3\times3\times3\times5\times5\times7\times13=5896800$$

is L. C. M. required.

This number 5896800 contains

224,	for it contains	$2\times2\times2\times2\times2\times7$
180,	,,	$2\times2\times3\times3\times5$
910,	,,	$2\times5\times7\times13$
1404,	,,	$2\times2\times3\times3\times3\times13$
2025,	,,	$3\times3\times3\times3\times5\times5$

and is therefore a common multiple. Again, no one of the factors can be spared, for if one of the twos were omitted, the result would not contain 224; the absence of one three would vitiate the result *as regards 2025,* and so on with the other prime factors.

The rule may be stated thus: To find L.C.M. of any numbers, 1st. Resolve each number into prime factors. 2nd. Select from these the highest power of each. 3rd. Find the continued product of these powers. If one of the given numbers is a measure of another of them, it may of course be disregarded. If all the numbers are prime to each other, their continued product is their L.C.M.

EXERCISE LXXI.

Find L. C. M. of:

(1) 2, 3, 5, 7
(2) 2, 3, 6
(3) 3, 5, 9, 25
(4) 3, 5, 15, 9, 25
(5) 6, 60, 12, 15, 20, 30
(6) 4, 8, 12, 16, 20
(7) 7, 14, 15, 21, 45
(8) 30, 40, 50, 60
(9) 16, 25, 81
(10) 80, 200, 45, 72, 225, 48
(11) 98, 35, 77, 121
(12) 26, 39, 52, 65

22. The method of the last paragraph may be shortened.

Find L. C. M. of 80, 200, 40, 45, 72, 225, 48, 36.

1st. Omit 40, 45 and 36, as being measures of 80, 225 and 72, respectively.

2nd. Take out the prime factor 5, common to 80, 200 and 225; divide each of them by 5, giving the series,

$\cancel{16}$, 40, 72, 45, 48.

Omit 16, a measure of 48. Again, taking out 5, we get the series,

8, 72, 9, 48.

Omit 8 and 9, measures of 72, leaving the two numbers 72 and 48. Taking out their G.C.M. 24, we get 2 and 3, prime to each other; hence $5 \times 5 \times 24 \times 3 \times 2 = 3600$ is L.C.M. required.

$$\begin{array}{r|llllllll}
5 & 80, & 200, & \cancel{40}, & \cancel{45}, & 72, & 225, & 48, & \cancel{36} \\
5 & \cancel{16}, & 40, & & & 3, & 45, & 2, & \\
24 & & 8, & & & & 9, & &
\end{array}$$

L.C.M. $= 5 \times 5 \times 24 \times 3 \times 2 = 3600$.

This method differs from that in § 21, only in that we take out the common factors from several numbers simultaneously. We may state the rule thus: To find L.C.M. of any numbers, 1st. Expel any of them which measure others. 2nd. Place on the left of the vertical line any prime factor common to two or more of the given numbers, and divide those numbers by it. 3rd. Perform these two

operations on the series thus obtained until we get a series of numbers all prime to one another. 4th. Find the continued product of the last series, and the factors on the left of the vertical line, which will be L. C. M. required.

N.B. We may bring out a composite instead of a prime factor when every member of the series is either divisible by or prime to this composite factor.

EXERCISE LXXII.

Find L. C. M. of:

(1) 24, 20, 18, 16, 12, 15
(2) 18, 36, 24, 35, 20
(3) 6, 10, 14, 15, 21, 35
(4) 30, 42, 105, 70
(5) 12, 20, 28, 18, 30, 42, 45, 63, 105, 70
(6) 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
(7) 21, 15, 33, 35, 77, 105, 165, 385, 231
(8) 16, 6, 8, 2, 12, 3, 48, 24
(9) 34, 26, 65, 85, 51, 39
(10) 10, 20, 30, 40, 50, 60
(11) 105, 120, 616, 88, 24, 12, 6, 308
(12) 12, 18, 27, 63, 28
(13) 7, 11, 4, 14, 10, 5, 15
(14) 323, 247, 209, 133
(15) 6, 5, 3, 11, 35, 44, 68, 17, 14
(16) 1003, 1357
(17) 899, 961
(18) 407, 703, 444
(19) 411, 959, 2055
(20) 120, 400, 500, 375, 1500, 1000, 960

(21) Three travellers journey 15, 18 and 24 miles a-day respectively. How far off is the first station at which all three put up?

(22) If the year of the planet Mercury were exactly 87, that of Venus 225, and of the earth 365 of our days; how many of our years would intervene between any two occasions on which the three planets would appear to a spectator from the sun to be in a straight line?

(23) If there be a house-door every 21 yds., and a lamp-post every 44 yds., 1 ft.; supposing a lamp-post exactly opposite one house-door, at what distance will the same occur again, and how many houses and lamp-posts will intervene?

23. Find the least number by which 35 must be multiplied to obtain a multiple of 56.

G. C. M. of 35 and $56 = 7$,
$\therefore$ L. C. M. of 35 and $56 = 5 \times 7 \times 8$.

This contains 7×8, 5 times, and 5×7, 8 times. Hence $8 \times 35 = 5 \times 56$. *Ans.* 8.

Find the least number of dollars at 4*s.* 2*d.* which is an exact number of crowns.

G. C. M. of 4*s.* 2*d.* and 5*s.* = 10*d.*
L. C. M. = 10*d.* $\times 5 \times 6$,
$\therefore$ $6 \times$ 4*s.* 2*d.* = $5 \times$ 5*s.* *Ans.* 6.

20	1080, 1260
9	54, 63
	6, 7

From this operation we can answer the following four questions:

(1) Find G.C.M. of 1080 and 1260. *Ans.* $9 \times 20 = 180$.

(2) Find L.C.M. of 1080 and 1260. *Ans.* $6 \times 7 \times 180 = 7560$.

(3) Find the least number by which 1080 must be multiplied to yield a multiple of 1260.

$7 \times 1080 = 6 \times 1260$. *Ans.* 7.

(4) Find the least number by which 1260 must be multiplied to yield a multiple of 1080.

$6 \times 1260 = 7 \times 1080$. *Ans.* 6.

Exercise LXXIII.

(1) Answer the above four questions in the same order with regard to the following pairs of numbers:

a. 600 and 480. *d.* 1109 and 1867.
b. 1564 and 1932. *e.* 500 and 729.
c. 2530 and 1760. *f.* 2574 and 3336.

(2) What is the smallest sum of money that can be expressed, either

a. as guineas or pounds;
b. as crowns or half-guineas;
c. as a multiple of 15*s.* 9*d.* or of 17*s.* 6*d.*?

(3) Find the smallest weight which can be expressed by an exact number either of lbs. troy or lbs. avoirdupois.

(4) If 1 lb. of sugar is worth $5\frac{1}{2}$*d.*, and 1 lb. of coffee, 1*s.* 2*d.*; find the smallest number of lbs. of coffee which is worth an exact number of lbs. of sugar.

(5) Find the value of the smallest number of lbs. of coffee at 1*s.* 3*d.* that can be exchanged for an exact number of lbs. of tea at 2*s.* 9*d.* each.

(6) Of two cog-wheels with 75 and 120 teeth respectively, a particular tooth of the smaller wheel comes in contact with a tooth of the larger. In how many turns of each wheel will these two teeth meet again, and how many contacts will there have been?

24. It is required to mix two qualities of wine at £1. 15*s.* and £2. 10*s.* per gallon, so as to produce a quality worth £2. 2*s.* per

gallon. Find the least number of gallons of each that will produce the required mixture.

£2. 2s. − £1. 15s. = 7s.
£2. 10s. − £2. 2s. = 8s.

∴ on each gallon at £1. 15s. there is a gain of 7s.
on each gallon at £2. 10s. there is a loss of 8s.

and as the losses must be compensated by the gains, our problem requires us to take the least number of gallons on which there is a gain of 7s. that shall balance an exact number of losses of 8s. Now $7 \times 8s. = 8 \times 7s.$ ∴ 8 gallons at £1. 15s. + 7 gallons at £2. 10s. will yield the required mixture.

Verification:

	£	s.	d.
$8 \times$ £1. 15s. =	£14	0	0
$7 \times$ £2. 10s. =	17	10	0
∴ 15 gallons cost	31	10	0

∴ 1 gallon costs £31. 10s. ÷ 15 = £2. 2s.

Mix two qualities of goods at 24s. 6d. and 31s. $7\frac{1}{2}$d. per cwt., so as to yield a quality worth 26s. $4\frac{1}{2}$d. per cwt. Required the smallest number of cwts. of each.

Profit, 1s. $10\frac{1}{2}$d.; loss, 5s. 3d.

$1\frac{1}{2}$d.	1s. $10\frac{1}{2}$d.,	5s. 3d.
3	15,	42
	5,	14

∴ $14 \times$ 1s. $10\frac{1}{2}$d. = $5 \times$ 5s. 3d., and ∴ 14 cwts. of the cheaper added to 5 cwts. of the dearer quality will yield 19 cwts. of the required mixture.

Exercise LXXIV.

Mix (using the least number of the smallest units given in each component):

(1) lbs. at $5\frac{1}{2}$d.	with	lbs. at 3d.	to yield	lbs. at 4d.
(2) lbs. at 1s. 8d.	„	lbs. at 2s. 1d.	„	lbs. at 1s. $10\frac{1}{2}$d.
(3) lbs. at 3s. $4\frac{1}{2}$d.	„	lbs. at 1s. $10\frac{1}{2}$d.	„	lbs. at 2s. 3d.
(4) Casks at £1. 3s. $9\frac{3}{4}$d.	„	casks at 14s. $9\frac{3}{4}$d.	„	casks at £1. 1s.
(5) Tons at £15. 2s. 6d.	„	cwts. at £1. 11s. 3d.	„	cwts. at £1.
(6) lbs. at $5\frac{1}{2}$d.	„	lbs. at 1s. $2\frac{1}{4}$d.	„	lbs. to be sold at 1s. per lb., making a profit of $2\frac{3}{4}$d. per lb.

(7) A has to travel 1800 miles in Tartary. On foot he can travel 16 miles a day, on horseback 40 miles a day. For how many days can he afford to walk so as to complete his journey in 75 days?

(8) Mix gallons of spirits at 26s. 3d. per gallon with water, so as to sell the mixture at 18s. per gallon.

(9) Mix gallons of spirits at £1. 1*s*. 3*d*. per gallon with water, so as to sell the mixture for 12*s*. 8*d*. per gallon, and make a profit of 2*s*. 8*d*. per gallon.

25. When there are more than two given qualities, the problem becomes indeterminate, as we may mix any quantities of the qualities dearer than the proposed price of the mixture, and also any quantities of the cheaper qualities, and find the price per unit of each of these two mixtures; the problem is then reduced to that of mixing *two* qualities only.

26. ON THE TESTS BY CASTING OUT NINES AND ELEVENS.

In Chapter III. § 7, it is said of the test by casting out nines, that "if the results do not agree, the answer cannot possibly be correct; but if they do agree, the chances are at least eight to one in favour of its correctness." This implies that the nine test is not infallible: as a matter of fact, the nine test tells us that the result to be anticipated is some multiple of 9 + a particular number; thus if the remainder after casting out nines be 5, the result to be anticipated must be one of the numbers occurring in the series,

$$5,\ 5+9,\ 5+18,\ 5+27,\ \&c.,$$
$$\text{i.e. } 5,\quad 14,\quad 23,\quad 32,\quad \&c.,$$

all of which differ from one another by a multiple of nine; hence, if there be an error which the test fails to detect, this error must be a multiple of nine.

Similarly, an error not detected by the eleven test must be a multiple of eleven, for the result to be anticipated is some multiple of eleven + a particular number (say 7). Hence, the result must be some number occurring in the series,

$$7,\ 7+11,\ 7+22,\ 7+33,\ \&c.,$$
$$\text{i.e. } 7,\quad 18,\quad 29,\quad 40,\quad \&c.,$$

all of which differ from one another by some multiple of eleven.

An error not detected by either test must be a common multiple of nine and eleven, i.e. a multiple of ninety-nine. Suppose the nine test were to claim as a result a number occurring in the series 7, 16, 25, &c., and the eleven test a number in the series 3, 14, 25, &c., the result must occur in both series. By trial it is found that the first number occurring in both (being the first of the second series

the sum of whose digits is 7) is 25; hence the result must occur in the series 25, 25 + 99, 25 + 198, &c.

The nine test will not reveal an error caused by displacement of digits, but the eleven test often will. For example:

$$\begin{array}{ll} \dot{1}\dot{5}\dot{7}\dot{5}\dot{9}\dot{3}. \times 17 & \\ \hline 2679081 & (a) \end{array} \qquad \begin{array}{ll} \dot{1}\dot{5}\dot{7}\dot{5}\dot{9}\dot{3} \times 71 & \\ \hline 11189103 & (b) \end{array}$$

$$\begin{array}{ccc} & 6 & \\ 3 & \times & 8 \\ & 6 & \end{array} \quad \begin{array}{ccc} & 9 & \\ 7 & \times & 6 \\ & 9 & \end{array} \qquad \begin{array}{ccc} & 6 & \\ 3 & \times & 8 \\ & 6 & \end{array} \quad \begin{array}{ccc} & 2 & \\ 7 & \times & 5 \\ & 2 & \end{array}$$

$$\begin{array}{ll} \dot{1}\dot{5}\dot{7}\dot{5}\dot{9}\dot{3}.. \times 107 & \\ \hline 16862451 & (c) \end{array} \qquad \begin{array}{ll} \dot{1}\dot{5}\dot{7}\dot{5}\dot{9}\dot{3} \times 701 & \\ \hline 110472693 & (d) \end{array}$$

$$\begin{array}{ccc} & 6 & \\ 3 & \times & 8 \\ & 6 & \end{array} \quad \begin{array}{ccc} & 1 & \\ 7 & \times & 8 \\ & 1 & \end{array} \qquad \begin{array}{ccc} & 6 & \\ 3 & \times & 8 \\ & 6 & \end{array} \quad \begin{array}{ccc} & 1 & \\ 7 & \times & 8 \\ & 1 & \end{array}$$

If the product (*a*) had been required, and (*b*) or (*c*) or (*d*) obtained erroneously, the nine test would have failed to detect the error, but the eleven test would have detected it. Again: if the product (*c*) had been required, and (*a*) or (*b*) had been erroneously obtained, the nine test would again have failed to detect the error, and the eleven test would have been efficient. If (*d*) had been obtained erroneously, both tests would have allowed the error to pass.

Division is liable to a kind of error which neither test can possibly detect, viz. the final remainder may erroneously be greater than the divisor. All that the tests tell us is, that divisor × quotient + remainder = dividend, which may still hold, though the remainder be greater than the divisor.

27. Divisibility of £. *s*. *d*.

A sum of money is divisible by 2 if the number of farthings is 0 or 2, because any number of pence, shillings and pounds are each divisible by 2.

A sum of money is divisible by 4 if there are no farthings, because pence, &c., are, but farthings are not, divisible by 4.

A sum of money is divisible by 8 if there are no farthings and the number of pence is even, because pounds and shillings are each divisible by 8, and pence only if they are even.

A sum of money is divisible by 3 if the pence and farthings

together are divisible by 3, because pounds and shillings are each divisible by 3.

A sum of money is divisible by 6 if it is divisible both by 2 and by 3. (See § 8, *g.*)

A sum of money is divisible by 12 if it is divisible both by 3 and by 4. (See § 8, *i.*)

A sum of money is divisible by 5 if the shillings (omitting half-sovereigns), pence and farthings together are divisible by 5, because sovereigns and half-sovereigns are divisible by 5.

A sum of money is divisible by 10 if it is divisible by 2 and by 5.

To ascertain if a sum of money is divisible by 9, cast out nines from the pounds, add twice the remainder to the shillings, cast out nines from the result, add three times the remainder to the pence, cast out nines from this result, add four times the remainder to the farthings, and cast out nines; if there is no remainder the sum of money is divisible by 9, and if not, not.

Take, for example, £2838. 14*s.* $9\frac{3}{4}$*d.*

Wording: 2, 10, 1, 4, 12, 3; 6, 20, 2; 6, 15, 6; 27, 0′.

The sum is divisible by 9.

Again: Cast out nines from £2756. 18*s.* $4\frac{1}{4}$*d.*

Wording: 2, 9, 0, 5, 11, 2; 4, 22, 4; 12, 16, 7; 29, 2′.

If £2756. 18*s.* $4\frac{1}{4}$*d.* be divided by 9, the remainder will be 2 farthings = $\frac{1}{2}$*d.*

The reason for the process is as follows: £1 = 20 shillings = 2×9 shillings + 2 shillings. Rejecting the nines, we have 2*s.* from every pound, similarly 1*s.* = 12 pence = 9*d.* + 3*d.*; rejecting the nines, we have 3*d.* from every shilling, and 1*d.* = 4 farthings. Hence, as stated above, we multiply successively by 2, 3 and 4.

N.B. If the sum of money is not divisible by 3, which can be readily seen, it cannot be divisible by 9.

To ascertain whether a sum of money is divisible by 11, multiply successively by 9, 1 and 4.

Take, for example, £53562. 13*s.* 4*d.*

Wording: 2, 7, 1, 3; 27, 5, 18, 7; 11, 0′.

The sum is divisible by 11.

Again : Cast out elevens from £1723. 16*s.* $8\frac{1}{2}$*d.*

Wording: 6, 4, 7; 63, 8, 24, 2; 10, 42, 9′.

If £1723. 16*s.* $8\frac{1}{2}$*d.* be divided by 11, the remainder will be 9 farthings = $2\frac{1}{4}$*d.*

The reasoning for this is analogous to that for divisibility by 9.

CHAPTER XII.

MISCELLANEOUS EXAMPLES.

1. (1) If 4 articles cost £5. 7*s.* 8*d.*, what will a dozen cost?

(2) If 6 articles cost 17*s.* $3\frac{1}{2}$*d.*, what will 30 cost?

(3) If 20 men can do a piece of work in 17 days, how long will 4 men take?

(4) If 7 silk handkerchiefs cost £1. 8*s.* $10\frac{1}{2}$*d.*, what will 91 cost?

(5) If a dozen of wine cost £2. 2*s.*, what will 3 bottles cost?

(6) If 28 lbs. cost £7. 17*s.* 6*d.*, what will 4 lbs. cost?

(7) Find the cost of 5 articles at £6. 6*s.* 8*d.* per score.

(8) If 17 articles cost £3. 18*s.* $7\frac{1}{2}$*d.*, what will 1 cost?

(9) Find the cost of 15 articles, if 8 cost £1. 7*s.* 4*d.* [Here first find the cost of 1 article, and thence of 15.]

(10) Find the cost of 35 articles, if 15 cost £7. 4*s.* $5\frac{1}{4}$*d.* [15 and 35 not being prime to one another (G.C.M. 5), it is shorter and easier to find the value of 5, and thence of 35.]

(11) If the provisions of a fortress will last 60 days, allowing each man 24 oz. per day, how long will the provisions last, if the allowance is reduced to 15 oz.?

(12) And what may be the daily allowance, for the fortress to hold out 144 days?

(13) Find the cost of 9 articles, if 8 cost £17. 4*s.* 10*d.*

(14) Find the cost of 25 articles, if 45 cost £3. 10*s.* $3\frac{3}{4}$*d.*

(15) If 1 cwt. cost £30, what will 77 lbs. cost?

(16) If 1 ton cost £8. 8*s.* 4*d.*, what will 12 cwt. cost?

(17) If for 20 shillings I can travel 150 miles, how far shall I be able to travel for 12*s.*?

(18) If 2 tons, 5 cwt. cost £23. 12*s.* 6*d.*, what will 1 ton, 5 cwt. cost?

(19) If in 1 year, 8 months, I put by £45, how much shall I put by in $2\frac{1}{2}$ years?

(20) If 1 cwt., 2 qrs., 12 lbs. cost £3. 15*s.* 9*d.*, what will 1 cwt., 3 qrs., 4 lbs. cost?

2. Brackets.

Simplify:

(1) $(8+3)\times 5+10$
(2) $(145+29+10)\times 6+100$
(3) $(2519 - 728)\times 45+512$
(4) $2519\times 45 - 728\times 45+512$
(5) $(358+119)\times 7+99$
(6) $99+(119+358)\times 7$
(7) $99+119+358\times 7$
(8) $(99+119+358)\times 7$
(9) $99\times 7+119\times 7+358\times 7$
(10) $67\times 19+25$
(11) $(67+25)\times 19$
(12) $67\times 19+67\times 8$
(13) $67\times(19+8)$
(14) $48520\times 1976+48520\times 1090$
(15) $48520\times(1976+1090)$
(16) $(58512+7426)+(58512 - 7426)$
(17) $(58512+7426) - (58512 - 7426)$

[Note that to (16) the answer is twice the greater number, and to (17) twice the less. Why?]

(18) $758\times 758 - 757\times 757$
(19) $(125796+18043+237509+2759286)\div 602$
(20) $(106033+112568)\div(45+62)$
(21) $(106033+112568)\div(437+598+612+396)$
(22) $(436\times 436 - 157\times 157)\div(436 - 157)$
(23) $(8424\times 7056\times 102)\div 9072$
(24) $(3492\times 2049\times 867)\div 15606$
(25) $(34936\times 816\times 2046)\div 209616$
(26) (£1. 12*s.* $8\frac{1}{4}$*d.* $\times 63\times 112)\div$ £1. 1*s.* $9\frac{1}{2}$*d.*
(27) (£1. 12*s.* $8\frac{1}{4}$*d.* $\times 63\times 112)\div 392$

(28) (£597. 13*s*. 11*d*. × 845) — (£597. 13*s*. 11*d*. × 843)
(29) 28519 × 63 — 28519 × 53
(30) 62743 × 1509 — 62740 × 1509
(31) (14688 × 1045 × 10110) ÷ 108851
(32) (812345 + 109876 + 234567 + 321098 + 456789 + 159576) ÷ (3257 × 643)
(33) £14. 7*s*. $3\frac{3}{4}$*d*. × 79 + £14. 7*s*. $3\frac{3}{4}$*d*. × 21

3. (1) London contains 400778 houses, each inhabited by 8 persons on an average. Find the total number of inhabitants.

(2) The water-works of London supply 26 gallons a-day for each person. How much is supplied in a year to all London?

(3) How much will this water weigh, at 10 lbs. per gallon?

(4) In 1868 the births were in London 115744, and the deaths 74908. What was the increase of the population from this source?

(5) The area of the metropolis is 77997 acres. How many square miles is this? (1 square mile = 640 acres.)

(6) The total revenue of the United Kingdom in 1868 was £71,860,677. 12*s*. 8*d*., and the expenditure was £74,082,280. 5*s*. 5*d*. Find the deficit.

(7) The different sources of inland revenue yielded in 1868: excise, £20,173,288; stamps, £9,461,010; taxes, £3,450,318; income-tax, £6,184,166. The income from the same sources in 1867 was £39,159,781. Find the increase or decrease.

(8) The income-tax was levied at 6*d*. in the pound. Find the amount of income taxed.

(9) Divide £15,000 among A, B, C and D, giving to A £1961. 0*s*. 8*d*. more than to either of the others, who have equal shares.

(10) At an election, the Liberal candidate obtained 1375 votes more than the Conservative. The total number of votes polled was 7209. How many voted for the Conservative?

(11) A leaves an estate of £100,000, of which he disposes as follows: building lodging-houses, £1750; endowment of parish school, 1000 guineas; hospitals, £500; to the church, £500; to each of 4 chapels, £125; to each of 7 old servants, 19 guineas; the residue to be divided amongst his family, giving one-third to his widow, one-fourth to his daughter, and the balance in equal shares to his 2 sons. Find their respective shares.

(12) Find the total weekly wages of 325 men, 108 women and 75 children, who receive 4*s*. 3*d*., 2*s*. 10*d*., 1*s*. 1*d*. per day, for each man, woman and child respectively.

(13) A exchanges with B 1 cwt., 11 lbs. of coffee, at 10½*d*. per lb., for 5 cwt., 1 qr., 12 lbs. of sugar, at 3½*d*. per lb. The difference is to be paid in money. How much is to be paid, and by whom?

(14) Find the average age of 7 persons, aged respectively 47, 38, 52, 45, 41, 49, 43 years. [An average is the sum of a number of quantities divided by the number of these addenda.]

(15) Find the average height of the following Peaks: Monte Viso, 12580 ft.; Genèvre, 11780 ft.; Cenis, 11457 ft.; Izéran, 13266 ft.; M. Blanc, 15732 ft.; Matterhorn, 14835 ft.; Rosa, 15150 ft.; Gallenstock, 12475 ft.; Vogelberg, 10866 ft.; Ortlerspitz, 12852 ft.; Groszglockner, 12776 ft.; Finster-Aarhorn, 14109 ft.; Jungfrau, 13176 ft.

(16) A mixes 4 gallons of spirit at 10*s*. 9¾*d*., with 6 gallons at 16*s*. 1½*d*., and 5 gallons at 7*s*. 6*d*. each. Find the average cost per gallon.

(17) Find the average age of a school in which there are 20 boys at 9 years old, 4 at 10, 10 at 11, 12 at 12, 11 at 13, 2 at 14, and 1 at 15?

(18) A piece of translation consists of 32 pages, averaging 21 lines of 15 words each. What will be the charge at the rate of 3*s*. 6*d*. for every 72 words? Also at the rate of 3*s*. 6*d*. for every 96 words?

(19) I mixed 2 cwt., 2 qrs., 20 lbs., at 6½*d*. per lb., with 4 cwt., 2 lbs., at 4*d*. per lb. Find the price per lb. of the mixture.

(20) What is my income, if at 7*d*. in the £, I pay £11. 1*s*. 8*d*. income-tax?

(21) £11. 1*s*. 8*d*. ÷ 7.

(22) A's income is 500 guineas. What will be left him after paying the income-tax of 5*d*. in the £?

(23) How many times will a coach wheel, of 13 ft., 9 in. in circumference, turn round in going from London to Brighton, 50 miles?

(24) In exchange for 833 articles at 1*s*. 4*d*. each, I gave 39 guineas and 100 articles. What was the cost of each of the latter?

(25) Divide £100. 16*s.* among A, B, C and D, giving to A £10. 10*s.* more than to B, and to B £5. 5*s.* more than to either C or D, who have equal shares.

(26) I spent £97. 1*s.* 8*d.* on equal quantities of three kinds of goods, at 4*s.* 5*d.*, 6*s.* 3*d.* and 8*s.* 9*d.* each article respectively. How many articles did I buy?

(27) Divide £171. 10*s.* among 5 men, 6 women and 7 boys, giving to each woman twice a boy's share, and to each man three times a woman's share.

(28) Bought half a ton of sugar for £15, and sold it at $4\frac{1}{4}d.$ per lb. Find profit or loss.

(29) Find the least number of rupees at 2*s.* 3*d.* each that shall also be an exact number of rupees at 1*s.* $10\frac{1}{2}d.$ each.

(30) What is the greatest number by which 7927 and 8773 can be divided, leaving remainders 80 and 100 respectively?

(31) Bought 24 yards of cloth for £4. 3*s.* For how much must the whole be sold to gain $6\frac{3}{4}d.$ per yard?

(32) I sold 243 sheep at £2. 7*s.* 6*d.* each, and with the proceeds bought as many oxen at $16\frac{1}{2}$ guineas each as my money would allow. How many oxen did I buy, and what was over?

(33) How many francs at 10*d.* each can I get for £58. 12*s.* 6*d.*?

(34) Find the cost of 1 ton, 5 cwt., 16 lbs., at $3\frac{3}{4}d.$ per lb.

(35) A railway, 27 miles in length, is estimated to cost £15000 per mile. How many shares at £25 each must be issued?

(36) Find a number of pounds between £365 and £380, which is also an exact number of guineas.

(37) Find all the sums of money between £280 and £300, which are multiples both of 6*s.* 3*d.* and of 11*s.* 3*d.*

(38) Prove (*a*) that the G. C. M. of any two numbers can never exceed their difference; (*b*) that any two consecutive numbers must be prime to each other; (*c*) that if two numbers are divided by their G. C. M. the quotients are prime to each other; (*d*) that any number which is divisible by two other numbers will be divisible by their L. C. M.; (*e*) that one-third of the difference between any number and the sum of its digits is divisible by 3; (*f*) that every prime

number but 2 can be made composite by the addition or subtraction of unity; (*g*) that every prime number greater than 3 can be made a multiple of 6 either by the addition or else by the subtraction of unity; (*h*) that any two consecutive odd numbers must be prime to each other.

(39) Express 28437 in the undecimal and duodecimal scales.

(40) If I buy 20 gross of pens at 9*d*. per dozen, and sell them at 1*d*. each, what profit do I make?

(41) If I mix 50 gallons of spirit at 14*s*. 3*d*. per gallon, with 64 gallons of water, at what price per gallon must I sell the mixture to gain £8. 1*s*. 6*d*.?

(42) If my salary is 100 guineas per annum, what should I be paid from June 3rd to October 27th?

(43) If my salary is £300 a-year, how much a-year should I lose by being paid £5. 15*s*. per week; and how much should I gain by being paid £5. 15*s*. 6*d*.?

(44) 33 tons of coal, bought at 23*s*. per ton, are sold at 1*s*. 6*d*. per cwt. Find total profit.

(45) Required the weight of 17 boxes, each weighing 2 cwt., 17 lbs.

(46) How many men would weigh a ton, if they weigh on an average 10 stone (of 14 lb.) each?

(47) How many fathoms are there in 17 m., 6 fur., 90 yards?

(48) How many weeks have there been from the beginning of the 19th century to January 6th, 1869, counting leap-years?

(49) If I mix of four different drugs, 5 drs., 2 scr., 14 grs.; 1 oz., 3 drs., 2 scr., 17 grs.; 2 oz., 7 drs., 19 grs., and 6 oz., 4 drs. respectively, and make up the mixture into 26 doses, what will each dose weigh?

(50) If I subscribe 3 guineas the first year, and increase my subscription by 10*s*. 6*d*. each successive year, how much shall I have given in 10 years?

(51) If I begin with £768. 9*s*. 9*d*., and spend each month one-third of what I have at the beginning of that month, what will be left me after 6 months?

(52) If the Liberal majority in 1868 was 65, and in 1869 was 119, how many seats must have been won?

(53) A sovereign weighs 123 grains; how many can be coined out of 41 oz. troy?

(54) What would 21000 sovereigns weigh in avoirdupois weight?

(55) How long would a velocipede take over 50 miles, at the rate of a furlong a minute?

(56) If a box holds 20 bags of corks, each holding a gross, what will 503 boxes cost at $4\frac{1}{2}d.$ per dozen corks?

(57) The four quarters of the year 1869 begin as follows: March 20th, 1 h., 32 m., p.m.; June 21st, 10 h., 4 m., a.m.; September 23rd, 28 m. a.m.; December 21st, 6 h., 23 m., p.m. Find the lengths of spring, summer and autumn; and taking the year as 365 d., 5 h., 48 m., find the length of winter.

(58) If with $8\frac{1}{2}$ dozen oranges at $1\frac{1}{2}d.$ each, and $31\frac{1}{2}$ lbs. sugar at $5\frac{1}{2}d.$ per lb., I make 45 pots of marmalade, what is the cost per pot?

(59) I spent £291. 5*s.* on equal quantities of 3 different kinds of goods, costing respectively 13*s.* 3*d.*, 18*s.* 9*d.*, £1. 6*s.* 3*d.* each article. How many of each kind did I buy?

(60) How many doses of 5 dwt., 8 grs., can be made of 3 lbs. troy?

(61) A dealer bought 560 sheep at £2. 4*s.* 6*d.* each, and 320 oxen at £18. 10*s.* each. He sold 160 sheep at £2. 18*s.* each, and the remainder at £2. 5*s.* each. Of the oxen, he sold 45 at £20 each, and the remainder at $18\frac{1}{2}$ guineas each. His expenses in the transaction were £37. 10*s.* Did he gain or lose by the transaction, and how much?

(62) Make out a bill for the following purchases:

				s.	*d.*	
12	lbs. of	mutton	@	0	$8\frac{1}{2}$	per lb.
$6\frac{1}{2}$	,,	,,	@	0	9	,,
$14\frac{1}{2}$	,,	beef	@	0	$11\frac{1}{2}$	,,
$8\frac{1}{4}$	,,	,,	@	0	8	,,
$5\frac{3}{4}$	,,	pork	@	1	0	,,
$10\frac{1}{2}$	,,	cheese	@	0	$9\frac{1}{2}$	,,

(63) Find the cost of constructing a railroad 125 miles long, at the rate of 16 guineas per yard.

(64) What do you understand by the prime factors of a number? How may you determine by the inspection of the digits of a number when it is divisible by the numbers 2, 3 and 11 respectively. Find G.C.M. of 7854 and 9768.

(65) Reduce 167805 lbs. avoirdupois to tons.

(66) Find all the divisors of 2145.

(67) I bought 8027 articles at 5*s.* 10$\frac{1}{4}$*d.* each, and sold 4000 articles at 6*s.* 3*d.* each. At how much a-piece must I sell the remainder to make a profit of £309. 17*s.* 7$\frac{1}{4}$*d.* on the whole?

(68) Explain the terms: Measure, prime numbers, odd numbers, and numbers prime to each other.

(69) What sense, if any, can you attach to the following expressions:

a. £15. 3*s.* 8*d.* + £3. 11*s.* 9*d.*
b. £15. 3*s.* 8*d.* — £3. 11*s.* 9*d.*
c. £15. 3*s.* 8*d.* × £3. 11*s.* 9*d.*
d. £15. 3*s.* 8*d.* ÷ £3. 11*s.* 9*d.*
e. £15. 3*s.* 8*d.* × £1.
f. £15. 3*s.* 8*d.* × 0
g. £15. 3*s.* 8*d.* × 1

(70) How many coins, each worth 12*s.* 7*d.*, must be given in exchange for 143 coins at 16*s.* 10$\frac{1}{2}$*d.* each, added to 567 coins at 10*s.* 1$\frac{1}{2}$*d.* each?

(71) Find all the common measures of 5082, 9438, 10890 and 8712.

(72) If of a series of quotients obtained by dividing each of a given series of dividends by a common measure, one quotient is prime to all the others, the common divisor is the G.C.M. of the series of dividends. Prove it.

(73) If 50 shares are bought at £35. 10*s.* 6*d.* each, and sold for £2042. 18*s.* 4*d.*, what is the gain per share?

(74) A company has 2000 shares of £50 each. It fails, and its debt, £58333. 6*s.* 8d. must be paid by the shareholders. How much will the holder of 73 shares have to pay, and what will be his total loss if he bought the shares at half-price?

(75) How long would light, which travels from the sun to the earth in 8 minutes, take from the nearest fixed star, which is 200000 times as far as the sun?

(76) A train started from London at 9.15 a.m., and reached Bristol, 120 miles off, at 12.25 p.m.; it stopped at 5 stations, losing 5 minutes at each, with 15 minutes extra at Swindon for refreshment. Find the average rate of the train.

(77) Divide £6842. 14*s.* 5*d.* among A, B and C, so that A may have £568. 14*s.* 4*d.* more than B, and C £728. 18*s.* 2*d.* less than B.

(78) Find G.C.M. of 15 h. 12 min., and 1 d., 3 h., 33 min.

(79) State and prove the test of accuracy by casting out of nines in division.

[(80) Find the criteria of divisibility by 2, 3, 4 and 5 of numbers expressed in the senary scale.

(81) If the number of odd digits in any number expressed in a scale with an odd radix be even, the number is divisible by 2; if not, not.]

(82) Three watches are set together; the first gains 6, the second 8, and the third 10 minutes a-week. In how many weeks will they again shew the same time?

(83) Convert £5013 to guineas, and 5013 guineas to pounds.

(84) Is 1109 a prime number? Describe the shortest way of deciding the question.

(85) Find G.C.M. of 16776 and 2096, and explain each step of the process. Also of 12018, 20030, and 30045.

(86) Define the Least Common Multiple, and find L.C.M. of 85, 125, 1445, 4913.

(87) Find the cost of 4 tons, 13 cwts., 1 qr., 19 lbs., at $2\frac{3}{4}d.$ per lb., and of 2000 oz. troy, at £3. 17*s.* $10\frac{1}{2}d.$ per oz.

(88) If a certain number of yards at 1*s.* $1\frac{1}{4}d.$ per yard, and the same number of yards at 1*s.* $5\frac{1}{2}d.$ per yard, amount together to £515. 1*s.* 3*d.*, how many yards are there of each?

(89) If 450 articles at £2. 10*s.* 6*d.* per article, and a certain number of articles at 5*s.* $8\frac{1}{2}d.$ each, amount together to £1215. 6*s.* $2\frac{1}{2}d.$, how many articles of the latter kind are there?

(90) If 45 oxen at £18. 12*s*. 6*d*. each, and 75 sheep, together cost £971. 5*s*., how much does each sheep cost?

(91) A fast train starts $2\frac{1}{2}$ hours after a slow one; in what time will it overtake the latter, their rate being 42 and 30 miles per hour respectively?

(92) A and B start from York and London, travelling 23 miles and 17 miles per hour respectively. In what time will they meet, and what distance from London, the total distance being 200 miles?

(93) A house and its furniture together are worth £2085, the value of the house being 4 times that of the furniture. Find the cost of each.

(94) Find the mean of the following observations: $43^\circ\ 15'$, $48^\circ\ 13'$, $42^\circ\ 17'$, $47^\circ\ 1'$, $50^\circ\ 50'$, $46^\circ\ 19'$, 51°, $44^\circ\ 10'$, $35^\circ\ 12'$, $38^\circ\ 47'$.

(95) Express 4113 (decimal) in the quinary and duodecimal scales.

(96) Convert *tetet* from the duodecimal into the senary scale.

(97) I bought 701 articles for £14493. 3*s*. 6*d*., and sold them at a profit of £8. 6*s*. 6*d*. each; the total proceeds I invested in some mining shares, each costing £87. 12*s*. 6*d*. and yielding £3. 8*s*. 9*d*. a-year. Find my total yearly income.

(98) The fore wheel of a carriage is 8 feet, 9 inches in circumference; the hind wheel 14 feet, 7 inches. If a nail on the outside of each wheel touch the ground at starting, how many times in the course of a mile will the same two nails be on the ground together?

(99) At the Crystal Palace were admitted on a Foresters' day 83500 persons, each paying 1*s*. How many admissions on a half-crown day would amount to an equal sum?

(100) A person after paying an income-tax of 5*d*. in the £, has £979. 3*s*. 4*d*. left. Find his gross income.

WEIGHTS AND MEASURES.

MEASURES OF LENGTH AND SURFACE.

Lineal Measure.

12 inches (in.)...............	= 1 foot (ft.)
3 feet........................	= 1 yard (yd.)
1 fathom	= 2 yards = 6 feet
5½ yards	= 1 rod or pole (po.)
40 poles, or 220 yards ...	= 1 furlong (fur.)
8 furlongs, or 1760 yards	= 1 mile (m.)

Gunter's Chain (used for Land-measuring).

100 links, 22 yards, or 4 poles......	= 1 chain (ch.)
484 square yards	= 1 sq. chain
10 sq. chains, or 100,000 sq. links	= 1 acre (ac.)
80 chains	= 1 mile (m.)

Cloth Measure (used by Drapers, Mercers, Clothiers, &c.).

2¼ inches (in.) ...		= 1 nail (nl.)
9 ,,	(4 nails)	= 1 quarter (qr.)
36 ,,	(4 qrs.)	= 1 yard (yd.)

SQUARE MEASURE.

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet...............	= 1 square yard (sq. yd.)
30¼ square yards............	= 1 square pole (sq. pl.)
40 square poles	= 1 rood (ro.)
4 roods (4840 yards)......	= 1 acre (ac.)
640 acres	= 1 square mile (m.)

MEASURES OF VOLUME.

Solid or Cubic Measure.

1728 cubic inches	= 1 cubic foot (c. ft.)
27 cubic feet...	= 1 cubic yd. (c. yd.)

APOTHECARIES WEIGHT.

20 grains troy	= 1 scruple
3 scruples	= 1 drachm
8 drachms	= 1 ounce (troy)
12 ounces.....	= 1 pound (troy)

MEASURES OF WEIGHT.

Avoirdupois Weight (used in almost all commercial transactions).

16 drams (dr.)...........	=	1 ounce (oz.)
16 ounces..	=	1 pound (lb.) = 7000 grains
28 lbs....................	=	1 quarter (qr.)
4 quarters, or 112 lbs.	=	1 hundred-weight (cwt).
20 hundred-weight......	=	1 ton (ton)

Troy Weight (used in weighing Gold and Silver, &c.).

24 grains	=	1 pennyweight (dwt.)
20 pennyweights	=	1 ounce (oz.)
12 ounces.........	=	1 pound (lb.) = 5760 grains

MEASURES OF CAPACITY.

Liquid.

4 gills......	=	1 pint (pt.)
2 pints ...	=	1 quart (qt.)
4 quarts...	=	1 gallon (gal.)

Dry Measure.

2 pints (pts.)	=	1 quart (qt.)
4 quarts......	=	1 gallon (gal.)
2 gallons ...	=	1 peck (pk.)
4 pecks	=	1 bushel (bus.)
8 bushels ...	=	1 quarter (qr.)
5 quarters...	=	1 load (ld.)

ANGULAR MEASURE.

60 seconds (″)	=	1 minute (1′)
60 minutes...	=	1 degree (1°)

MEASURES OF TIME.

60 seconds	=	1 minute (min.)
60 minutes	=	1 hour (hr.)
24 hours	=	1 day (d.)
12 calendar months	=	1 year (yr.)
365 days..............	=	1 year (yr.)

PAPER MEASURE.

24 sheets	=	1 quire (qr.)
20 sheets	=	1 outside quire (out. qr.)
20 quires	=	1 ream (rm.)
10 reams	=	1 bale (bl.)

Printed by C. Green & Son, 178, Strand.

THE SCIENCE AND ART

OF

ARITHMETIC;

For the Use of Schools.

PART II. VULGAR FRACTIONS.
PART III. APPROXIMATE CALCULATIONS.

BY

A. SONNENSCHEIN

AND

H. A. NESBITT, M.A., UNIV. COLL., LONDON.

"The mills of God grind slowly, but they grind exceeding small."

SECOND EDITION, REVISED AND ENLARGED.

FOURTH THOUSAND.

LONDON:
WHITTAKER AND CO.

1875.

PREFACE TO FIRST EDITION.

The Second and Third Parts of our Arithmetic, bound together in the present Volume, treat of Fractional quantities in a two-fold aspect. In the Second Part the results are obtained by means of Vulgar Fractions, and are, often at the cost of much labour, "needlessly accurate." In the Third Part, calculations are made with Decimal Fractions, which, in the large majority of cases, save a great deal of labour by systematically disregarding minute quantities. We have thus the power of obtaining accuracy to within any assigned degree, which is all that is needed in the affairs of life. To use, however, such methods without lapsing into slovenliness of thought, it has been necessary to introduce at once the notion of Limits. It is evident that in so elementary a treatise as this, hardly more could be done than indicate the existence in the human mind of such notions; but not only is no harm done, but great good is gained, by opening vistas of thought to the young student, and by carefully impressing upon him that in science there is no such thing as finality.

This consideration also must be our justification both for the introduction and for the superficial treatment of Converging Fractions (Part III. Chap. I.): for their introduction, because they furnish the easiest conception of Approximation; for their superficial treatment, because a knowledge of Algebra is required for their thorough comprehension, and they have on the whole very little bearing on the ultimate objects of the book. This Chapter may therefore be omitted at discretion.

Teachers are often in doubt as to the proper sequence of subjects to be taught. It has been suggested that Decimals ought to be taught before Vulgar Fractions, being so much more easy to manipulate. This theory is very tempting, but fails in the cardinal point that the notion of Decimals is a much later one than that of Vulgar Fractions. The binary divisions are in fact the earliest with which the human mind is acquainted. Moreover, the operations of Decimals are mere applications of rules previously established in Vulgar Fractions. Who, for example, could realize that $\cdot 2 \times \cdot 3 = \cdot 06$, if he had not previously understood multiplication by Vulgar Fractions? The principle of appealing to the experience of the student in order to establish the rule, has induced us to interweave mental arithmetic exercises in the earlier Chapters on Vulgar Fractions. Proportion is treated as a later notion of Fractions, which in fact it is, so long as we confine ourselves to numbers.

The methods we give for the ready Decimalization of English Money, Weights, and Measures, are, with a slight exception, entirely new. They yield, we believe, all the advantages to be derived from the introduction of Decimal Coins, Weights and Measures, without incurring the fearful inconveniences of a change, or forfeiting the very notable benefits conferred by the subdivision of the English sovereign into $20 \times 12 \times 4$ farthings. As long as there are twelve months in the year, and as long as men prefer to deal in dozens rather than in tens, so long will 12 be a most convenient factor in our coin. Whether our weights and measures ought to be superseded in favour of a decimal system, may be fairly considered an open question; but till such change is effected, the methods of Part III. Ch. VIII., will very nearly answer the same purpose.

A. SONNENSCHEIN.
H. A. NESBITT.

SEPT. 3RD, 1870.

PREFACE TO SECOND EDITION.

In the present Edition we have made a few alterations, notwithstanding the disadvantages that always attend the introduction of changes into a School Book in general use. We had, in fact, to choose between the evil of stagnation and the inconvenience of improvement.

These changes (disregarding mere corrections) are:

I. The Notion of Reciprocals has been more fully developed.

II. A separate Chapter has been introduced on the "Unitary Method" immediately after Division of Vulgar Fractions.

III. A part of the text on Ratio and Proportion has been rewritten, notional reasoning having been substituted for symbolic reasoning.

IV. Several sections in Part III. have been transposed to earlier stages, viz.:

(*a*) Simplification of Complex Decimals and G.C.M. and L.C.M. of Decimals from Ch. XII. to Ch. III.

(*b*) Number of Decimal Places in a Continued Product from Ch. XII. to Ch. IV., with introduction of the two new Exercises XXI. (*b*) and XXIII. (*b*).

(*c*) Compound Interest from Ch. XII. to Ch. X.

V. The Table of Contents has been amplified.

Of these changes, only the second requires comment. We have effected a complete severance between the two modes of thought;

i.e. the more primitive mode of conscious distinct reference to Unity as an intermediate step, and the more advanced method of reasoning on relative magnitude, the reference to Unity being less prominent in the mind; this last is an indispensable step towards the conception of ratios between incommensurable quantities.

Some writers, indeed, advocate the omission of "Proportion" altogether from Arithmetic, recommending the sole use of the Unitary Method, which they call "Reasoning by First Principles." We contend, however, that there is no special virtue in primitive methods, and that, on the contrary, they are necessarily cumbrous. All progress is towards simplicity. It is no doubt necessary that the student should be able to fall back upon Elementary Reasoning, but he must not abide by it. We might, on the plea of "First Principles," advocate the rejection of Multiplication and Division in favour of Cumulative Addition and Subtraction. In a word, all contractions of operations entail the losing sight of "First Principles," and progress in any science would be impossible if the earlier truths had to be reiterated at each new step.

By the early introduction of the "Unitary Method" there is gained the practical advantage that pupils are enabled to solve "Rule of Three" questions as soon as possible, which is a decided convenience in other branches of study, such as Chemistry, Natural Philosophy, &c.

A. SONNENSCHEIN.
H. A. NESBITT.

APRIL, 1875.

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PART II. VULGAR FRACTIONS.

PART II.

VULGAR FRACTIONS.

ARITHMETIC.

Part II. VULGAR FRACTIONS.

CHAPTER I.

UNIFORM DENOMINATORS.

§ 1. The word Fraction is derived from the Latin *fractus*, broken, and Fractional differs from Integral Arithmetic in that it deals with parts of things instead of with whole things; but, as we shall see, the same notions of More and Less still apply. It is to be understood that the parts here spoken of are only such as are obtained by dividing the whole into *equal* parts.

§ 2. If a whole is divided into two equal parts, each of them is called *one-half;* and therefore there are two halves in a whole.

If a whole is divided into three equal parts, each part is called *one-third*, and therefore there are three thirds in a whole.

If a whole is divided into four equal parts, each part is called *one-fourth*, or *one-quarter*, and therefore there are four fourths or quarters in a whole.

Similarly a whole may be divided into seven sevenths, twenty-one twenty-firsts, thirty-two thirty-seconds, &c.

Exercise I.

Find one-half, one-third, one-quarter, one-fifth, one-sixth, one-seventh, one-eighth, one-ninth, one-tenth, one-eleventh and one-twelfth of £57. 15*s.*

§ 3. One-half, one-third, &c., are written, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, &c.

Two-thirds of a whole are obtained by dividing the whole into three equal parts, and taking two of them.

Three-quarters of a whole are obtained by dividing the whole into four equal parts, and taking three of them.

Five-sevenths of a whole are obtained by dividing the whole into seven equal parts, and taking five of them, and so on.

These parts are written, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{7}$, &c.

The lower figure of every fraction, then, shews us into how many equal parts the whole has been divided, and might therefore be called the *Divisor;* but by shewing us the number of parts into which the whole has been divided, it also shews us the *Name* of each part, and is therefore called the NAMER, or, more commonly, the DENOMINATOR.

The upper figure of every fraction shews us *how many* of these parts we take, and is therefore called the COUNTER, or, more commonly, the NUMERATOR.

N.B. The German words *Zähler* and *Nenner* give the same ideas in more familiar language. In German, the ordinal *fünfte* is clearly distinguished from the fractional *Fünftel;* but unfortunately in English the word *fifth* has both these meanings, which sometimes requires to be pointed out.

The words Numerator and Denominator are derived from the Latin *numerare,* to count, and *nominare,* to name.

Example. Find $\frac{2}{3}$ of £157. 4*s.* $5\frac{1}{4}$*d.*

$$\frac{1}{3} \text{ of } £157.\ 4s.\ 5\tfrac{1}{4}d. = 3\underline{)157 \quad 4 \quad 5\tfrac{1}{4}}$$
$$52 \quad 8 \quad 1\tfrac{3}{4}$$
$$2$$
$$\frac{2}{3} \text{ of } \quad ,, \quad = £104 \quad 16 \quad 3\tfrac{1}{2}$$

EXERCISE II.

	£.	s.	d.		£.	s.	d.
Find (1) $\frac{2}{3}$ of	197	15	$4\frac{1}{2}$	Find (7) $\frac{4}{9}$ of	8043	17	3
(2) $\frac{3}{4}$ of	1037	17	6	(8) $\frac{7}{10}$ of	27419	12	$8\frac{1}{2}$
(3) $\frac{2}{5}$ of	23	7	$9\frac{3}{4}$	(9) $\frac{5}{11}$ of	489	18	$11\frac{1}{4}$
(4) $\frac{5}{6}$ of	1001	15	0	(10) $\frac{11}{12}$ of	3584	16	3
(5) $\frac{3}{7}$ of	893	1	$9\frac{3}{4}$	(11) $\frac{7}{15}$ of	23	7	$9\frac{3}{4}$
(6) $\frac{5}{8}$ of	347	13	6	(12) $\frac{11}{18}$ of	65	17	9

EXERCISE A.*

(1) A journey is divided into two equal parts; what will each part be called? *Ans.* One half of the journey.

(2) A goose weighs 7 lbs. and half its own weight; what is the weight of the goose? *Ans.* 14 lbs.

* *All Exercises* numbered alphabetically are to be worked mentally.

(3) A post is buried one half in the ground, and there are five feet above ground; what is the length of the post? *Ans.* 10 feet.

(4) A post is driven through the water into the ground below; $\frac{1}{3}$ is in the ground, $\frac{1}{3}$ under water, and 10 feet above the water. What is the length of the whole post? *Ans.* 30 feet.

(5) Divide the half of 6*d.* among 3 children; what will each child get? *Ans.* 1*d.*

(6) How much is $\frac{2}{3}$ of 1*s.*? *Ans.* 8*d.*

(7) Out of 6*d.* I spend $\frac{1}{3}$ of 1*s.*; how much is left me? *Ans.* 2*d.*

(8) Find the difference between $\frac{2}{3}$ of 1*s.* and $\frac{1}{2}$ of 1*s.* *Ans.* 2*d.*

(9) From $\frac{3}{4}$ of £1 take $\frac{1}{2}$ of 1*s.* *Ans.* 14*s.* 6*d.*

(10) From $\frac{3}{4}$ of £1 take $\frac{2}{3}$ of a guinea. *Ans.* 1*s.*

(11) I sold $\frac{5}{6}$ of a dozen of wine; how many bottles were left? *Ans.* 2 bottles.

(12) To $\frac{3}{5}$ of £1 add $\frac{5}{6}$ of 1*s.* *Ans.* 12*s.* 10*d.*

(13) How much is $\frac{2}{5}$ of an hour? *Ans.* 24 minutes.

(14) How would you get $\frac{7}{11}$ of a thing?
Ans. Divide the whole into 11 equal parts and take 7 of them.

(15) If I cut off $\frac{4}{7}$ of a thing, what part of the whole will be left? *Ans.* $\frac{3}{7}$ of the whole.

(16) Distribute $\frac{4}{7}$ of a guinea among 6 persons. *Ans.* 2*s.* each.

(17) How much is $\frac{5}{12}$ of a yard? *Ans.* 1 ft. 3 in.

Exercise III.

(1) Find the length of $\frac{4}{5}$ of a mile.

(2) Divide $\frac{7}{11}$ of a mile into 8 equal parts.

(3) Find $\frac{4}{7}$ of a ton.

(4) Find $\frac{3}{8}$ of 1 lb. troy.

(5) Find the value of £$\frac{4}{5}$ + £$\frac{2}{3}$ + $\frac{5}{6}$ of 1*s.*

(6) Find the value of $\frac{2}{3}$ of a guinea + £$\frac{1}{3}$ — $\frac{3}{4}$ of 1*d.*

(7) Distribute $\frac{5}{7}$ of 1 cwt. of coals among four persons.

(8) Divide the distance of $\frac{5}{11}$ of 7 miles into 8 equal portions.

(9) If I spend $\frac{2}{3}$ of a guinea per day, how much is that in 10 days?

(10) Find the value of the whole amount, if $\frac{4}{5}$ of it is £429. 6*s.* 8*d.*

(11) Find the value of £$\frac{1}{2}$, £$\frac{1}{3}$, £$\frac{2}{3}$, £$\frac{1}{4}$, £$\frac{3}{4}$, £$\frac{1}{5}$, £$\frac{2}{5}$, £$\frac{3}{5}$, £$\frac{4}{5}$, £$\frac{1}{6}$, £$\frac{5}{6}$, £$\frac{1}{8}$, £$\frac{3}{8}$, £$\frac{5}{8}$, £$\frac{7}{8}$, £$\frac{1}{10}$, £$\frac{1}{12}$, £$\frac{1}{15}$, £$\frac{1}{16}$, £$\frac{1}{20}$.

(12) Find the value of $\frac{1}{2}$ of 1*s.*, $\frac{1}{3}$ of 1*s.*, $\frac{2}{3}$ of 1*s.*, $\frac{1}{4}$ of 1*s.*, $\frac{3}{4}$ of 1*s.*, $\frac{1}{6}$ of 1*s.*, $\frac{5}{6}$ of 1*s.*, $\frac{1}{8}$ of 1*s.*, $\frac{3}{8}$ of 1*s.*, $\frac{5}{8}$ of 1*s.*, $\frac{7}{8}$ of 1*s.*, $\frac{1}{12}$ of 1*s.*

§ 4. We know that $\frac{4}{5}$ means a whole divided into 5 equal parts, of which we take 4; similarly $\frac{5}{5}$ means a whole divided into 5 parts, of which we take all 5, that is, $\frac{5}{5}$ is the whole; but what is the meaning of $\frac{6}{5}$? We must evidently extend our definition of a fraction. $\frac{6}{5}$ must mean that *more wholes than one* are divided into 5 equal parts *each*, and that we take 6 such parts.

$\frac{2}{2}=\frac{3}{3}=\frac{4}{4}=\frac{5}{5}=\frac{6}{6}$, &c. $=1$;
$\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$ are each less than unity;
$\frac{5}{5}=$ unity, and $\frac{6}{5}$, $\frac{7}{5}$, $\frac{8}{5}$, &c., are each greater than unity; hence,

If the numerator is less than the denominator, the value of the fraction is less than unity.

If the numerator equals the denominator, the value of the fraction equals unity.

If the numerator is more than the denominator, the value of the fraction is more than unity.

Fractions which are less than unity are called *Proper Fractions.*

Fractions which are equal to or more than unity are called *Improper Fractions.*

Quantities consisting of an integer *and* a fraction are called *Mixed Numbers*; e.g. $3\frac{1}{2}$, $5\frac{2}{3}$, $114\frac{17}{19}$. (These are read three and a half; five and two-thirds, &c.)

EXERCISE B.

(1) How many halves in $1\frac{1}{2}$?	*Ans.* 3 halves.
(2) How many halves in $5\frac{1}{2}$?	„ 11 halves.
(3) How many thirds in $3\frac{2}{3}$?	„ 11 thirds.
(4) How many thirds in 5 wholes?	„ 15 thirds.
(5) How many quarters in $2\frac{3}{4}$?	„ 11 quarters.
(6) How many wholes in 10 halves?	„ 5 wholes.
(7) How many halves in 10 wholes?	„ 20 halves.
(8) How many wholes in 17 halves?	„ $8\frac{1}{2}$.
(9) How many wholes in 17 thirds?	„ $5\frac{2}{3}$.
(10) How many wholes in 20 fifths?	„ 4 wholes.

(11) Reduce to whole or mixed numbers:

$\frac{13}{5}$.	*Ans.* $2\frac{3}{5}$.	$\frac{25}{8}$.	*Ans.* $3\frac{1}{8}$.	$\frac{35}{9}$.	*Ans.* $3\frac{8}{9}$.
$\frac{15}{4}$.	,, $3\frac{3}{4}$.	$\frac{29}{7}$.	,, $4\frac{1}{7}$.	$\frac{35}{10}$.	,, $3\frac{5}{10}$.
$\frac{11}{9}$.	,, $1\frac{2}{9}$.	$\frac{30}{10}$.	,, 3.	$\frac{35}{12}$.	,, $2\frac{11}{12}$.
$\frac{17}{5}$.	,, $3\frac{2}{5}$.	$\frac{30}{11}$.	,, $2\frac{8}{11}$.	$\frac{35}{3}$.	,, $11\frac{2}{3}$.

(12) Reduce to improper fractions:

$9\frac{3}{8}$.	*Ans.* $\frac{75}{8}$.	$8\frac{4}{7}$.	*Ans.* $\frac{60}{7}$.	$6\frac{7}{8}$.	*Ans.* $\frac{55}{8}$.
$10\frac{4}{5}$.	,, $\frac{54}{5}$.	$5\frac{7}{12}$.	,, $\frac{67}{12}$.	$11\frac{2}{3}$.	,, $\frac{35}{3}$.
$6\frac{4}{11}$.	,, $\frac{70}{11}$.	$10\frac{7}{13}$.	,, $\frac{137}{13}$.	$4\frac{4}{9}$.	,, $\frac{40}{9}$.
$5\frac{4}{5}$.	,, $\frac{29}{5}$.	$4\frac{5}{6}$.	,, $\frac{29}{6}$.	$3\frac{2}{3}$.	,, $\frac{11}{3}$.

(13) I distributed seven apples among some children, giving to each $\frac{1}{3}$ of an apple. How many children were there?

Ans. 21 children.

(14) Which is greater, and by how much, $\frac{7}{5}$ or 2 wholes?

Ans. 2 wholes are greater by $\frac{3}{5}$.

(15) If I cut up $8\frac{2}{5}$ yards of tape into strips of $\frac{1}{5}$ yard each, how many strips shall I get? *Ans.* 42 strips.

§ 5. Reduce the mixed number $8\frac{5}{13}$ to an improper fraction. One whole $=\frac{13}{13}$; eight wholes $=\frac{104}{13}$, and the $\frac{5}{13}$ make $\frac{109}{13}$.

Exercise IV.

Reduce to improper fractions:

(1) $2\frac{1}{2}$.	(6) $84\frac{17}{20}$.	(11) $41\frac{583}{1000}$.	(16) $5000\frac{83}{5000}$.
(2) $3\frac{1}{3}$.	(7) $864\frac{13}{97}$.	(12) $41\frac{83}{1000}$.	(17) $100\frac{28}{33}$.
(3) $7\frac{2}{3}$.	(8) $46\frac{7}{10}$.	(13) $41\frac{3}{1000}$.	(18) $10000\frac{14}{15}$.
(4) $8\frac{3}{4}$.	(9) $46\frac{17}{100}$.	(14) $41\frac{3}{10000}$.	(19) $3001\frac{83}{700}$.
(5) $12\frac{13}{15}$.	(10) $46\frac{7}{100}$.	(15) $400\frac{127}{400}$.	(20) $73\frac{72}{99}$.

Exercise C.

(1) How many wholes in 3 halves? *Ans.* $1\frac{1}{2}$.
(2) ,, ,, 5 halves? ,, $2\frac{1}{2}$.
(3) ,, ,, 5 thirds? ,, $1\frac{2}{3}$.
(4) ,, ,, 5 quarters? ,, $1\frac{1}{4}$.
(5) ,, ,, 7 quarters? ,, $1\frac{3}{4}$.
(6) ,, ,, 11 quarters? ,, $2\frac{3}{4}$.
(7) ,, ,, 11 fifths? ,, $2\frac{1}{5}$.

(8) How many wholes in 15 fifths? *Ans.* 3.

(9) Find the difference between 5 wholes and 18 fifths. „ $1\frac{2}{5}$.

(10) If I spend $\frac{1}{7}$ of a shilling in a day, what shall I spend in 14 days? *Ans.* 2*s.*

(11) If I have 32-fifths of a cake, and wish to give a whole cake to each child, to how many children can I give it?

Ans. 6 children and $\frac{2}{5}$ of a cake over.

(12) Reduce to whole or mixed numbers:

$\frac{15}{2}$.	*Ans.* $7\frac{1}{2}$.	$\frac{65}{12}$.	*Ans.* $5\frac{5}{12}$.	$\frac{100}{10}$.	*Ans.* 10.
$\frac{19}{3}$.	„ $6\frac{1}{3}$.	$\frac{11}{10}$.	„ $1\frac{1}{10}$.	$\frac{109}{10}$.	„ $10\frac{9}{10}$.
$\frac{27}{4}$.	„ $6\frac{3}{4}$.	$\frac{27}{10}$.	„ $2\frac{7}{10}$.	$\frac{117}{10}$.	„ $11\frac{7}{10}$.
$\frac{37}{5}$.	„ $7\frac{2}{5}$.	$\frac{39}{10}$.	„ $3\frac{9}{10}$.	$\frac{128}{10}$.	„ $12\frac{8}{10}$.
$\frac{37}{6}$.	„ $6\frac{1}{6}$.	$\frac{48}{10}$.	„ $4\frac{8}{10}$.	$\frac{139}{10}$.	„ $13\frac{9}{10}$.
$\frac{37}{7}$.	„ $5\frac{2}{7}$.	$\frac{56}{10}$.	„ $5\frac{6}{10}$.	$\frac{139}{100}$.	„ $1\frac{39}{100}$.
$\frac{37}{8}$.	„ $4\frac{5}{8}$.	$\frac{65}{10}$.	„ $6\frac{5}{10}$.	$\frac{428}{100}$.	„ $4\frac{28}{100}$.
$\frac{40}{9}$.	„ $4\frac{4}{9}$.	$\frac{74}{10}$.	„ $7\frac{4}{10}$.	$\frac{4089}{100}$.	„ $40\frac{89}{100}$.
$\frac{45}{9}$.	„ 5.	$\frac{83}{10}$.	„ $8\frac{3}{10}$.	$\frac{4089}{1000}$.	„ $4\frac{89}{1000}$.
$\frac{50}{9}$.	„ $5\frac{5}{9}$.	$\frac{92}{10}$.	„ $9\frac{2}{10}$.	$\frac{4089}{4089}$.	„ 1.

§ 6. Reduce $\frac{299}{23}$ and $\frac{529}{29}$ to whole or mixed numbers. Since there are 23 twenty-thirds or 29 twenty-ninths in every whole, *as many times* as we can get 23 twenty-thirds out of 299 twenty-thirds, or 29 twenty-ninths out of 529 twenty-ninths, so many wholes will there be. Divide therefore the numerator by the denominator; the quotient will be wholes, and the remainder, if any, will be fractional parts with the same denominator as the proposed fraction.

23) 299 (13	29) 529 (18
69	239
..	7
$\frac{299}{23}=13$	$\frac{529}{29}=18\frac{7}{29}$

EXERCISE V.

Reduce to whole or mixed numbers:

(1) $\frac{19}{2}$.	(6) $\frac{529}{23}$.	(11) $\frac{158}{10}$.	(16) $\frac{648597}{10}$.
(2) $\frac{43}{3}$.	(7) $\frac{419}{21}$.	(12) $\frac{519}{10}$.	(17) $\frac{648597}{100}$.
(3) $\frac{59}{4}$.	(8) $\frac{584}{17}$.	(13) $\frac{8431}{10}$.	(18) $\frac{648597}{1000}$.
(4) $\frac{50}{5}$.	(9) $\frac{3519}{467}$.	(14) $\frac{12594}{10}$.	(19) $\frac{648597}{10000}$.
(5) $\frac{137}{8}$.	(10) $\frac{37}{10}$.	(15) $\frac{800000}{10}$.	(20) $\frac{648597}{100000}$.

§ 7. ADDITION AND SUBTRACTION OF FRACTIONS. Let us recal the meaning of the symbol $\frac{3}{4}$. We have said above that it means 3 pieces of equal size, each of which is called or *named* a quarter. Similarly, $\frac{4}{7}$ means 4 equal pieces, of which each is called a seventh. Note that the symbol 4 performs very different functions above and below the line. Above the line it represents quantity or *number;* below the line, *name.*

"In Addition and Subtraction we must always have the *same kind* of units, viz., so many things of one kind added to or taken from so many things of the *same* kind." (Part I. p. 59.) It follows that we can only add or subtract fractions of the same denomination.

EXERCISE D.

(1) $\frac{1}{2}+\frac{3}{2}$.	*Ans.* 2.	(11) $2\frac{2}{5}-\frac{4}{5}$.	*Ans.* $1\frac{3}{5}$.
(2) $\frac{2}{3}+\frac{2}{3}+\frac{1}{3}$.	„ $1\frac{2}{3}$.	(12) $1-(\frac{2}{5}+\frac{1}{5})$	„ $\frac{2}{5}$.
(3) $\frac{1}{4}+\frac{1}{4}+\frac{3}{4}$.	„ $1\frac{1}{4}$.	(13) $2\frac{3}{5}+3\frac{4}{5}$.	„ $6\frac{2}{5}$.
(4) $\frac{5}{2}+\frac{4}{2}$.	„ $4\frac{1}{2}$.	(14) $\frac{4}{5}+\frac{7}{5}$.	„ $2\frac{1}{5}$.
(5) $\frac{7}{2}-\frac{4}{2}$.	„ $1\frac{1}{2}$.	(15) $1\frac{2}{5}-\frac{4}{5}$.	„ $\frac{3}{5}$.
(6) $1\frac{1}{2}+2\frac{1}{2}$.	„ 4.	(16) $1\frac{7}{8}+\frac{3}{8}$.	„ $2\frac{2}{8}$.
(7) $1\frac{1}{3}+2\frac{2}{3}$.	„ 4.	(17) $1\frac{7}{8}-\frac{5}{8}$.	„ $1\frac{2}{8}$.
(8) $3\frac{1}{4}+\frac{3}{4}$.	„ 4.	(18) $2\frac{9}{10}-1\frac{7}{10}$.	„ $1\frac{2}{10}$.
(9) $2\frac{3}{4}+1\frac{1}{4}$.	„ 4.	(19) $3-2\frac{1}{10}$.	„ $\frac{9}{10}$.
(10) $4\frac{2}{3}-\frac{7}{3}$.	„ $2\frac{1}{3}$.	(20) $3\frac{3}{10}-1\frac{7}{10}$.	„ $1\frac{6}{10}$.

Simplify $\frac{9}{17}+\frac{3}{17}+\frac{14}{17}+\frac{11}{17}$. *Ans.* $\frac{37}{17}=2\frac{3}{17}$.

Simplify $8\frac{9}{17}+6\frac{3}{17}+5\frac{14}{17}+\frac{11}{17}$. *Ans.* $19\frac{37}{17}=19+2\frac{3}{17}=21\frac{3}{17}$.

It is generally shorter, however, to add the fractional parts first and carry the wholes; thus, in this last example, the fractions come to $\frac{37}{17}$ or $2\frac{3}{17}$; put down $\frac{3}{17}$ and carry 2 to the wholes. Similarly, in Subtraction we shall first deduct the fractional part, and then the wholes, but if the number of pieces in the subtrahend exceed that in the minuend, we must break up one of the wholes of the minuend, or else add the value of a whole to both terms, as we have done in subtraction of money. Thus:

$$9\tfrac{5}{8}-7=2\tfrac{5}{8};$$
$$9\tfrac{5}{8}-7\tfrac{3}{8}=2\tfrac{2}{8};$$
$$9\tfrac{5}{8}-7\tfrac{5}{8}=2;$$
$$9-7\tfrac{5}{8}=8\tfrac{8}{8}-7\tfrac{5}{8}=1\tfrac{3}{8};$$

$$\text{or, } 9 - 7\tfrac{5}{8} = 9\tfrac{8}{8} - 8\tfrac{5}{8} = 1\tfrac{3}{8};$$
$$9\tfrac{1}{8} - 7\tfrac{5}{8} = 8\tfrac{9}{8} - 7\tfrac{5}{8} = 1\tfrac{4}{8};$$
$$\text{or, } 9\tfrac{1}{8} - 7\tfrac{5}{8} = 9\tfrac{9}{8} - 8\tfrac{5}{8} = 1\tfrac{4}{8}.$$

EXERCISE VI.

(1) $2\frac{1}{2} + 3\frac{1}{2} + 5\frac{1}{2}$.

(2) $8\frac{1}{3} + 7\frac{1}{3} + 15\frac{2}{3} + 1\frac{1}{3}$.

(3) $42\frac{3}{4} + 58\frac{2}{4} + 3\frac{1}{4} + 19\frac{3}{4} + 11\frac{1}{4}$.

(4) $73\frac{4}{5} + 5\frac{2}{5} + 7\frac{1}{5} + 184\frac{3}{5} + 2049\frac{2}{5}$.

(5) $5\frac{13}{17} + 8\frac{16}{17} + 12\frac{11}{17} + 1\frac{9}{17}$.

(6) $3\frac{5}{12} + 52\frac{7}{12} + 87\frac{1}{12} + 943\frac{11}{12}$.

(7) $278\frac{15}{28} - 189$.

(8) $278\frac{15}{28} - 189\frac{15}{28}$.

(9) $278\frac{15}{28} - 189\frac{11}{28}$.

(10) $278 - 189\frac{11}{28}$.

(11) $278\frac{11}{28} - 189\frac{15}{28}$.

(12) $52\frac{3}{10} + 86\frac{1}{10} + 42\frac{9}{10} + 5\frac{7}{10} + 84\frac{2}{10}$.

(13) $584\frac{37}{100} + 129\frac{9}{100} + 43\frac{27}{100} + 3\frac{93}{100} + 4\frac{87}{100} + 1\frac{99}{100} + 6\frac{7}{100}$.

(14) $1\frac{11}{100} + 6\frac{48}{100} + 11\frac{85}{100} + 17\frac{22}{100} + 22\frac{59}{100} + 27\frac{96}{100}$.

(15) $2\frac{43}{1000} + 3\frac{154}{1000} + 4\frac{265}{1000} + 5\frac{376}{1000} + 6\frac{487}{1000} + 7\frac{598}{1000} + 8\frac{709}{1000} + 1\frac{1}{1000}$.

(16) $2012\frac{16}{19} - 789$.

(17) $2012\frac{16}{19} - 789\frac{12}{19}$.

(18) $2012\frac{16}{19} - 789\frac{16}{19}$.

(19) $2012 - 789\frac{12}{19}$.

(20) $2012\frac{12}{19} - 789\frac{18}{19}$.

(21) $34\frac{17}{23} + 18\frac{11}{23} + 49\frac{16}{23} + 519\frac{20}{23} + \frac{21}{23}$.

(22) $4301\frac{10}{11} - 896$.

(23) $4301\frac{10}{11} - 896\frac{4}{11}$.

(24) $4301 - 896\frac{4}{11}$.

(25) $4301\frac{4}{11} - 896\frac{10}{11}$.

(26) $100\frac{4}{11} - (12\frac{7}{11} + 16\frac{8}{11} + 20\frac{9}{11} + 24\frac{10}{11})$.

(27) $(14\frac{3}{17} + 16\frac{8}{17} + 18\frac{13}{17} + 21\frac{1}{17} + 23\frac{6}{17}) - 10\frac{16}{17}$.

(28) $(5\frac{8}{100} + 11\frac{61}{100} + 18\frac{14}{100} + 24\frac{67}{100} + 31\frac{20}{100} + 37\frac{73}{100}) - (1\frac{1}{100} + 7\frac{54}{100} + 14\frac{7}{100} + 20\frac{60}{100} + 27\frac{13}{100} + 33\frac{66}{100})$.

(29) $(538\frac{123}{500} + 169\frac{329}{500}) + (538\frac{123}{500} - 169\frac{329}{500})$.

(30) $(538\frac{123}{500} + 169\frac{329}{500}) - (538\frac{123}{500} - 169\frac{329}{500})$.

(31) $\frac{16}{19}+\frac{16}{19}+\frac{16}{19}+\frac{16}{19}+\frac{16}{19}$.
(32) $7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}$.
(33) $8\frac{4}{5}+8\frac{4}{5}+8\frac{4}{5}+8\frac{4}{5}+8\frac{4}{5}$.

§ 8. Multiplication of Fractions by integers.

Exercise E.

(1) How much is 3 times $\frac{1}{2}$? *Ans.* $1\frac{1}{2}$.
(2) " 5 times $\frac{3}{2}$? " $7\frac{1}{2}$.
(3) " 8 times $\frac{3}{2}$? " 12.
(4) " 5 times $\frac{2}{3}$? " $3\frac{1}{3}$.
(5) " $\frac{1}{3}$ of $\frac{9}{4}$? " $\frac{3}{4}$.
(6) " $\frac{1}{3}$ of $4\frac{1}{2}$? " $1\frac{1}{2}$.
(7) " $\frac{1}{3}$ of $2\frac{1}{7}$? " $\frac{5}{7}$.
(8) " $\frac{1}{5}$ of $2\frac{1}{7}$? " $\frac{3}{7}$.
(9) How many times can we take $\frac{5}{7}$ out of $2\frac{1}{7}$? *Ans.* 3 times.
(10) " " $\frac{3}{7}$ out of $2\frac{1}{7}$? " 5 times.
(11) " " $1\frac{3}{4}$ out of $5\frac{1}{4}$? " 3 times.
(12) Divide $5\frac{1}{4}$ into 7 equal parts. *Ans.* $\frac{3}{4}$ to each part.
(13) Divide $7\frac{1}{5}$ into 6 equal parts. " $1\frac{1}{5}$ to each part.
(14) Repeat $\frac{2}{5}$ fourteen times. *Ans.* $5\frac{3}{5}$.
(15) What is the seventh part of $5\frac{3}{5}$? " $\frac{4}{5}$.
(16) Distribute $4\frac{4}{5}$ cakes among 6 boys.

Ans. $\frac{4}{5}$ of a cake to each boy.

(17) If I walk $2\frac{4}{5}$ miles an hour, how long will it take me to walk $8\frac{2}{5}$ miles? *Ans.* 3 hours.

(18) Take $\frac{4}{7}$ ten times. *Ans.* $5\frac{5}{7}$.
(19) Divide $5\frac{5}{7}$ into 10 equal parts. *Ans.* $\frac{4}{7}$ to each part.
(20) How many times is $\frac{4}{7}$ contained in $5\frac{5}{7}$? *Ans.* 10 times.
(21) Repeat $\frac{5}{8}$ eight times. *Ans.* 5.

(22) I distributed $8\frac{4}{7}$ cwt. of coals among some persons, giving to each $1\frac{5}{7}$ cwt. How many persons were there? *Ans.* 5 persons.

(23) If I spend $\frac{2}{7}$ of a shilling a-day, how much shall I spend in 6 days, and how long will 8*s.* last me?

Ans. $1\frac{5}{7}$*s.* in 6 days; 28 days.

(24) If 1 strip is $1\frac{2}{3}$ yards long, how many strips can I make of 10 yards? *Ans.* 6 strips.

(25) If I sell $\frac{2}{3}$ of a load of hay 8 times, how much do I sell? *Ans.* $5\frac{1}{3}$ loads.

(26) If one cap is made of $\frac{4}{5}$ of a yard, how many yards are wanted for 7 caps? *Ans.* $5\frac{3}{5}$ yards.

(27) If 1 coat requires $2\frac{3}{4}$ yards, how many coats can be cut out of $13\frac{3}{4}$ yards? *Ans.* 5 coats.

(28) How many strips of carpet, each $1\frac{2}{5}$ yards long, can be cut off a remnant $8\frac{2}{5}$ yards long? *Ans.* 6 strips.

(29) How many times will a wheel, $\frac{6}{7}$ of a yard in circumference, turn round in travelling over $12\frac{6}{7}$ yards? *Ans.* 15 times.

(30) How much ground will be travelled over by a wheel, $1\frac{3}{8}$ yards in circumference, after it has made 7 turns? *Ans.* $9\frac{5}{8}$ yards.

(31) How much ground will be travelled over by a wheel, $1\frac{3}{7}$ yards in circumference, after it has made $4\frac{1}{2}$ turns? *Ans.* $6\frac{3}{7}$ yards.

(32) Find the circumference of a wheel which makes 9 turns in travelling over $7\frac{1}{5}$ yards. *Ans.* $\frac{4}{5}$ of a yard.

Let $\frac{1}{F}$ be 1 farthing, $\frac{1}{HP} = 1$ halfpenny, $\frac{1}{P} = 1$ penny, $\frac{1}{S} = 1$ shilling.

$\frac{1}{F} \times 2 = \frac{2}{F} = \frac{1}{HP}$; again, $\frac{1}{P} \times 12 = \frac{12}{P} = \frac{1}{S}$; similarly, $\frac{7}{P} \times 12 = \frac{84}{P} = \frac{7}{S}$.

In each case here we have two answers, one obtained by alteration of the *number* of the things, and the other by alteration of the *name* of the things, keeping their number the same. In the same way,

$\frac{1}{8} \times 4 = \frac{4}{8} = \frac{1}{2}$ ∵* 1 whole $= \frac{8}{8}$.
$\frac{1}{15} \times 3 = \frac{3}{15} = \frac{1}{5}$ ∵ 1 whole $= \frac{15}{15}$.
$\frac{4}{15} \times 3 = \frac{12}{15} = \frac{4}{5}$, being 4 times as much as $\frac{1}{15} \times 3$.

Comparing the *two* answers to each of these questions, we again perceive that multiplication may be performed either by change of the *number* of the things or of the *name* of the things.

Examine the nature of each of these changes. When the number, i.e. the numerator, is changed, we *multiply*, but when the name, i.e. the denominator, is changed, we *divide*. These two operations cor-

* ∵ = because; ∴ = therefore.

respond to our previous notions of fractions. Thus, if we wish to treble any number of slices of a cake, we may either take three times as many slices, or make each slice three times as large. Suppose each slice to be $\frac{1}{15}$ of a cake; 3 times 4 such slices, or $\frac{4}{15} \times 3$, will be either 12 such slices, i.e. $\frac{12}{15}$, or 4 slices each 3 times as large as $\frac{1}{15}$; but 3 times $\frac{1}{15}$ is $\frac{3}{15} = \frac{1}{5}$, $\therefore$ the 4 slices will be $\frac{4}{5}$; hence $\frac{4}{15} \times 3 = \frac{12}{15}$ or $\frac{4}{5}$. Of these two answers, $\frac{4}{5}$ being the simpler is preferable.

$\frac{7}{16} \times 5 = \frac{35}{16}$. Here, 5 not being a measure of 16, we can at present apply only the first method.

Learn by heart: *To multiply a fraction by an integer, multiply the numerator or divide the denominator. Division is preferable where it can be done without remainder.* Thus:

$\frac{13}{20} \times 2 = \frac{13}{10} = 1\frac{3}{10}$.
$\frac{13}{20} \times 3 = \frac{39}{20} = 1\frac{19}{20}$.
$\frac{13}{20} \times 4 = \frac{13}{5} = 2\frac{3}{5}$.
$\frac{13}{20} \times 5 = \frac{13}{4} = 3\frac{1}{4}$.
$\frac{13}{20} \times 7 = \frac{91}{20} = 4\frac{11}{20}$, &c.
$3\frac{5}{12} \times 2 = 6\frac{5}{6}$.
$3\frac{5}{12} \times 3 = 9\frac{5}{4} = 10\frac{1}{4}$, or, beginning with the fraction, $\frac{5}{12} \times 3 = \frac{5}{4} = 1\frac{1}{4}$, put down $\frac{1}{4}$ and carry 1 whole: $3 \times 3 + 1 = 10$. *Ans.* $10\frac{1}{4}$.
$3\frac{5}{12} \times 4 = 13\frac{2}{3}$.
$3\frac{5}{12} \times 5 = 17\frac{1}{12}$, &c.

EXERCISE VII.

(1) $\frac{7}{18} \times 2$, 3, 4, 5, 6, 7, 8, 9.

(2) $\frac{11}{60} \times 2$, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 20.

(3) $\frac{59}{144} \times 2$, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 18, 24, 36, 72.

(4) $\frac{43}{100} \times 2$, 3, 4, 5, 7, 9, 10, 20, 25, 30, 50.

(5) $\frac{113}{240} \times 2$, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 20, 30, 40, 50, 60, 70, 80, 100, 120.

(6) $4\frac{1}{2} \times 2$, 3, 4, 5, 6, 7, 8, 9, 10.

(7) $11\frac{11}{30} \times 2$, 3, 4, 5, 6, 7, 8, 9, 10.

(8) $118\frac{419}{1000} \times 2$, 4, 5, 7, 11, 20, 24, 25, 100, 500.

(9) What will 9 men pay for their dinner, if each pays $\frac{3}{4}$ of a crown?

(10) If 1 horse eats $\frac{7}{8}$ of a load, what will 4 horses eat?

(11) If I consume $2\frac{5}{12}$ bushels a-week, how much shall I consume in 7 weeks? Also in 4 weeks?

(12) What will be paid for 6 articles at £$1\frac{11}{18}$ each?

§ 9. Division of Fractions by integers.

Exercise F.

(1) What is $\frac{1}{2}$ of $\frac{1}{2}$? *Ans.* $\frac{1}{4}$.

(2) How many thirds are there in a whole? *Ans.* 3 thirds.

(3) If each third is cut in half, how many of these smaller pieces will there be in the whole? *Ans.* 6 smaller pieces.

(4) What part of the whole would each smaller piece be? *Ans.* $\frac{1}{6}$.

(5) How much then is $\frac{1}{2}$ of $\frac{1}{3}$? " $\frac{1}{6}$.

(6) How many halves in a whole? *Ans.* 2 halves.

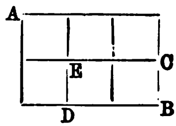
(7) If each half is divided into three equal parts, how many of these smaller pieces will there be in a whole? *Ans.* 6 smaller pieces.

(8) What part of the whole would each piece be? *Ans.* $\frac{1}{6}$.

(9) How much then is $\frac{1}{3}$ of $\frac{1}{2}$? " $\frac{1}{6}$.

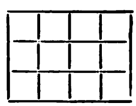
Teacher. Therefore $\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{3}$ of $\frac{1}{2} = \frac{1}{6}$.

Illustration:



AB is the whole, AC the half, AD the third, AE either half of the third, or one-third of the half, or one-sixth of the whole.

Similarly $\frac{1}{3}$ of $\frac{1}{4} = \frac{1}{4}$ of $\frac{1}{3} = \frac{1}{12}$:



(10) $\frac{1}{4}$ of $\frac{1}{5}$.	*Ans.* $\frac{1}{20}$.	(16) $\frac{1}{2} \div 3$.	*Ans.* $\frac{1}{6}$.
(11) $\frac{1}{5}$ of $\frac{1}{4}$.	" $\frac{1}{20}$.	(17) $\frac{1}{3} \div 2$.	" $\frac{1}{6}$.
(12) $\frac{1}{3}$ of $\frac{1}{8}$.	" $\frac{1}{24}$.	(18) $\frac{1}{3} \div 4$.	" $\frac{1}{12}$.
(13) $\frac{1}{8}$ of $\frac{1}{3}$.	" $\frac{1}{24}$.	(19) $\frac{1}{5} \div 4$.	" $\frac{1}{20}$.
(14) $\frac{1}{6}$ of $\frac{1}{10}$.	" $\frac{1}{60}$.	(20) $\frac{1}{10} \div 6$.	" $\frac{1}{60}$.
(15) $\frac{1}{2} \div 2$.	" $\frac{1}{4}$.	(21) $\frac{1}{8}$ of $\frac{1}{9}$.	" $\frac{1}{72}$.

(22) $\frac{1}{9}$ of $\frac{1}{8}$. *Ans.* $\frac{1}{72}$.
(23) $\frac{1}{9} \div 8$. " $\frac{1}{72}$.
(24) $\frac{1}{8} \div 9$. " $\frac{1}{72}$.
(25) $\frac{1}{7}$ of $\frac{1}{5}$. " $\frac{1}{35}$.
(26) $\frac{1}{7} \div 5$. " $\frac{1}{35}$.
(27) $\frac{1}{5}$ of $\frac{1}{7}$. " $\frac{1}{35}$.
(28) $\frac{1}{5} \div 7$. " $\frac{1}{35}$.
(29) $\frac{1}{5}$ of $\frac{1}{9}$. " $\frac{1}{45}$.
(30) $\frac{1}{9}$ of $\frac{1}{5}$. " $\frac{1}{45}$.
(31) $\frac{1}{8}$ of $\frac{1}{7}$. " $\frac{1}{56}$.
(32) $\frac{1}{7}$ of $\frac{1}{8}$. " $\frac{1}{56}$.
(33) $\frac{1}{4}$ of $\frac{1}{7}$. " $\frac{1}{28}$.
(34) $\frac{1}{7}$ of $\frac{1}{4}$. " $\frac{1}{28}$.
(35) $\frac{1}{9}$ of $\frac{1}{10}$. " $\frac{1}{90}$.
(36) $\frac{1}{10}$ of $\frac{1}{9}$. " $\frac{1}{90}$.
(37) $\frac{1}{5}$ of $\frac{1}{3}$. " $\frac{1}{15}$.
(38) $\frac{1}{3}$ of $\frac{1}{5}$. " $\frac{1}{15}$.
(39) $\frac{1}{10}$ of $\frac{1}{10}$. " $\frac{1}{100}$.
(40) $\frac{1}{8}$ of $\frac{1}{8}$. " $\frac{1}{64}$.
(41) $\frac{1}{9}$ of $\frac{1}{9}$. " $\frac{1}{81}$.
(42) $\frac{1}{5}$ of $\frac{1}{5}$. " $\frac{1}{25}$.
(43) $\frac{1}{10}$ of $\frac{1}{11}$. *Ans.* $\frac{1}{110}$.
(44) $\frac{1}{11}$ of $\frac{1}{10}$. " $\frac{1}{110}$.
(45) $\frac{1}{11}$ of $\frac{1}{11}$. " $\frac{1}{121}$.
(46) $\frac{1}{11}$ of $\frac{1}{12}$. " $\frac{1}{132}$.
(47) $\frac{1}{12}$ of $\frac{1}{11}$. " $\frac{1}{132}$.
(48) $\frac{1}{5} \div 9$. " $\frac{1}{45}$.
(49) $\frac{1}{9} \div 5$. " $\frac{1}{45}$.
(50) $\frac{1}{8} \div 7$. " $\frac{1}{56}$.
(51) $\frac{1}{7} \div 8$. " $\frac{1}{56}$.
(52) $\frac{1}{7} \div 4$. " $\frac{1}{28}$.
(53) $\frac{1}{4} \div 7$. " $\frac{1}{28}$.
(54) $\frac{1}{8} \div 3$. " $\frac{1}{24}$.
(55) $\frac{1}{3} \div 8$. " $\frac{1}{24}$.
(56) $\frac{1}{11} \div 12$. " $\frac{1}{132}$.
(57) $\frac{1}{12} \div 11$. " $\frac{1}{132}$.
(58) $\frac{1}{5} \div 8$. " $\frac{1}{40}$.
(59) $\frac{1}{10} \div 4$. " $\frac{1}{40}$.
(60) $\frac{1}{20} \div 2$. " $\frac{1}{40}$.
(61) $\frac{1}{2} \div 20$. " $\frac{1}{40}$.
(62) $\frac{1}{8} \div 5$. " $\frac{1}{40}$.
(63) $\frac{1}{10} \div 10$. " $\frac{1}{100}$.

$\frac{6}{\text{P}} \div 2$ means that 6 pence are to be distributed into 2 equal parts. *Ans.* $\frac{3}{\text{P}}$ to each part.

$\frac{1}{\text{P}} \div 2 = \frac{1}{\text{HP}}$; similarly, $\frac{5}{\text{P}} \div 2 = \frac{5}{\text{HP}}$; and $\frac{6}{\text{P}} \div 2 = \frac{6}{\text{HP}} = \frac{3}{\text{P}}$. In this last case, we have *two* answers, one obtained by alteration of the *number*, and the other by alteration of the *name*. In the same way, $\frac{8}{7} \div 2 = \frac{4}{7}$ to each part. $\frac{1}{7} \div 2 = ?$ Here we cannot divide the *number;* can we solve the question by altering the *name?* We have to divide a seventh into two equal parts, i.e. to find the half of $\frac{1}{7}$. *Ans.* $\frac{1}{14}$ to each part. Similarly, $\frac{5}{7} \div 2 = \frac{5}{14}$ to each part; also $\frac{8}{7} \div 2 = \frac{8}{14}$; but $\frac{8}{7} \div 2 = \frac{4}{7}$, therefore we obtain two answers, one by *dividing* the *numerator*, the other by *multiplying* the *denominator*. As before, these two operations correspond to our previous notions of fractions. Thus if we wish to distribute any number of slices of a cake among three persons, we may either distribute the slices, or break up each slice into three equal parts. Suppose each

slice to be $\frac{1}{7}$ of a whole cake; in dividing 6 such slices among 3 persons, we may either give 2 slices to each, i.e. $\frac{2}{7}$, or we may break up each seventh into 3 parts, and give to each person 6 such parts, of which each is $\frac{1}{3}$ of $\frac{1}{7}$, or $\frac{1}{21}$, therefore each person will have $\frac{6}{21}$. Of these two answers, $\frac{2}{7}$ being the simpler is preferable.

$\frac{5}{7} \div 3$. Here, 3 not being a measure of 5, we can only apply the method of breaking up the pieces. *Ans.* $\frac{5}{21}$.

Learn by heart: *To divide a fraction by an integer, either divide the numerator or multiply the denominator. Division is preferable where it can be done without remainder.*

$\frac{8}{9} \div 2 = \frac{4}{9}$ to each part.
$\frac{8}{9} \div 3 = \frac{8}{27}$,,
$\frac{8}{9} \div 4 = \frac{2}{9}$,,
$\frac{8}{9} \div 5 = \frac{8}{45}$,,
$\frac{8}{9} \div 8 = \frac{1}{9}$,,
$1\frac{5}{7} \div 2 = \frac{12}{7} \div 2 = \frac{6}{7}$ to each part.
$1\frac{5}{7} \div 3 = \frac{12}{7} \div 3 = \frac{4}{7}$,,
$1\frac{5}{7} \div 4 = \frac{12}{7} \div 4 = \frac{3}{7}$,,
$1\frac{5}{7} \div 5 = \frac{12}{7} \div 5 = \frac{12}{35}$,,
$1\frac{5}{7} \div 6 = \frac{12}{7} \div 6 = \frac{2}{7}$,,
$1\frac{5}{7} \div 7 = \frac{12}{7} \div 7 = \frac{12}{49}$,,
$1\frac{5}{7} \div 12 = \frac{12}{7} \div 12 = \frac{1}{7}$ to each part.
$14\frac{2}{7} \div 2 = 7\frac{1}{7}$ to each part.
$14\frac{2}{7} \div 3 = 4\frac{16}{21}$,,

Working: $3\underline{)14\frac{2}{7}}$
4 wholes and $2\frac{2}{7}$ or $\frac{16}{7}$ over; $\frac{16}{7} \div 3 = \frac{16}{21}$.

$14\frac{2}{7} \div 4 = 3\frac{4}{7}$ to each part.

Working: $4\underline{)14\frac{2}{7}}$
3 and $2\frac{2}{7}$ or $\frac{16}{7}$ over; $\frac{16}{7} \div 4 = \frac{4}{7}$.

$14\frac{2}{7} \div 5 = 2\frac{6}{7}$ to each part.

Working: $5\underline{)14\frac{2}{7}}$
2 and $4\frac{2}{7}$ or $\frac{30}{7}$ over; $\frac{30}{7} \div 5 = \frac{6}{7}$.

$14\frac{2}{7} \div 6 = 2\frac{16}{42}$ to each part.

Working: $6\underline{)14\frac{2}{7}}$
2 and $2\frac{2}{7}$ or $\frac{16}{7}$ over; $\frac{16}{7} \div 6 = \frac{16}{42}$.

EXERCISE VIII.

(1) $\frac{15}{16} \div 2, 3, 4, 5, 6, 7, 8.$
(2) $\frac{20}{21} \div 2, 3, 4, 5, 6, 7, 10.$
(3) $\frac{100}{107} \div 2, 3, 4, 5, 6, 10, 20, 100.$
(4) $5\frac{1}{4} \div 2, 3, 4, 5, 7, 10.$
(5) $38\frac{2}{5} \div 2, 3, 4, 5, 6, 7, 8, 12, 20.$
(6) $1423\frac{1}{17} \div 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$

§ 10. The two rules for multiplication and division of fractions by integers can be summarized thus: Either perform on the numerator the operation indicated by the sign, or on the denominator the opposite of what the sign indicates. In each case division is preferable to multiplication where it can be done without remainder.

EXERCISE IX.

(1) $\frac{144}{175} \times 2, 3, 4, 5, 6, 7, 8, 25, 35, 175.$
$\div 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 72.$
(2) $8\frac{74}{77} \times 2, 3, 4, 7, 11, 12, 77.$
$\div 2, 3, 4, 5, 6, 7, 10, 23, 230.$
(3) Find $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ of $\frac{12}{13}$.
(4) Find $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ of $860\frac{8}{11}$.
(5) A Hanoverian mile is $4\frac{13}{22}$ English miles nearly. Find the length in English miles of 11 Hanoverian miles.
(6) Find $\frac{2}{3}, \frac{3}{4}, \frac{3}{5}, \frac{5}{8}, \frac{7}{9}, \frac{7}{10}$ of $2522\frac{31}{72}$.
(7) How long will $13\frac{2}{7}$ pieces be, if 3 are $8\frac{2}{5}$ yards in length?
(8) Distribute $9\frac{1}{3}$ cwt. of potatoes among 7 families of 5 persons each. How much will be given to each family, and how much to each person?

(9) If 1 person consumes $\frac{4}{15}$ of a lb. a-day, how much will a family of 5 persons consume in a week?

(10) Find the average of the following lengths: $1\frac{4}{11}$ yds., $1\frac{6}{11}$ yds., $1\frac{7}{11}$ yds., $\frac{10}{11}$ yds., $1\frac{9}{11}$ yds.

(11) I empty into a vat 2 vessels of $4\frac{5}{7}$ gallons each, 3 vessels of $2\frac{6}{7}$ gals. each, 7 vessels of $1\frac{4}{7}$ gals. each, 5 vessels of $4\frac{1}{7}$ gals. each. Distribute the contents of the vat into 2 equal vessels; also into 3, 4, 5, 7, 12, and 29 equal vessels.

(12) If I walk $3\frac{5}{6}$ miles an hour, how far shall I walk in 2, 3, 4, 5, 6 hours?

§ 11. $\frac{1}{8} \times 8 = 1$ whole; therefore $\frac{3}{8} \times 8 = 3$ wholes, $\frac{7}{8} \times 8 = 7$ wholes. Similarly $\frac{1}{11} \times 11 = 1$ whole, $\frac{6}{11} \times 11 = 6$ wholes, &c. Hence:

Learn by heart: *Any fraction multiplied by its denominator gives for answer its numerator as a whole number.*

Exercise X.

(1) $\frac{5}{13} \times 13$.

(2) $\frac{19}{23} \times 23$.

(3) $\frac{2485}{9711} \times 9711$.

(4) $6\frac{4}{5} \times 5$.

(5) $10\frac{7}{19} \times 19$.

(6) $463\frac{61}{62} \times 62$.

(7) $8\frac{4}{7} \times 7, 14, 21$.

(8) $10\frac{5}{12} \times 24, 36, 60$.

Exercise G.

§ 12. (1) How many times are $\frac{3}{4}$ contained in $5\frac{1}{4}$? *Ans.* 7 times.

(2) If one cap requires $1\frac{3}{4}$ yards, how many can be made out of $5\frac{1}{4}$ yards? *Ans.* 3 caps.

(3) A distance of $5\frac{5}{11}$ miles is divided into lengths of $\frac{6}{11}$ of a mile each. How many such lengths are there? *Ans.* 10 lengths.

(4) By what number must $\frac{4}{9}$ be multiplied to give $4\frac{8}{9}$? *Ans.* By 11.

(5) How many times is $\frac{3}{8}$ contained in 9 wholes? *Ans.* 24 times.

(6) If one dinner cost $\frac{3}{5}$ of half-a-crown, how many dinners will 3 half-crowns pay for? *Ans.* 5 dinners.

(7) Interpret $8\frac{2}{5} \div \frac{2}{5}$.

Ans. How many times are $\frac{2}{5}$ contained in $8\frac{2}{5}$.

(8) $8\frac{2}{5} \div \frac{2}{5}$. *Ans.* 21 times.

(9) Interpret $4\frac{2}{7} \div \frac{6}{7}$ and $4\frac{2}{7} \div 6$.

Ans. The first means, How many times are $\frac{6}{7}$ contained in $4\frac{2}{7}$; the second means, Distribute $4\frac{2}{7}$ into 6 equal parts.

(10) What are the answers then?

Ans. 5 times; and $\frac{5}{7}$ to each part.

(11) If a man spends $\frac{3}{5}$ of his daily wages, how many days will it take him to accumulate 4 days' wages? *Ans.* 10 days.

(12) $(1\frac{3}{7} + \frac{6}{7}) \div (1\frac{3}{7} - \frac{6}{7})$. *Ans.* 4 times.

EXERCISE XI.

(1) $76\frac{12}{23} \div \frac{1}{23}, \frac{2}{23}, \frac{4}{23}, \frac{5}{23}, \frac{8}{23}, \frac{11}{23}, \frac{16}{23}$.

(2) $76\frac{12}{23} \div 3\frac{11}{23}, 4\frac{18}{23}, 7\frac{15}{23}, 19\frac{3}{23}$.

(3) If a wheel is $2\frac{6}{7}$ feet in circumference, how many turns will it make in travelling over $17\frac{1}{7}$ feet?

(4) How many times is $\frac{5}{17}$ contained in $4\frac{12}{17}$?

(5) To how many people can I give $\frac{11}{21}$ of a load each out of $5\frac{5}{21}$ loads?

EXERCISE XII.

(1) If I consume $2\frac{5}{8}$ bushels in one week, how much shall I consume in 3 days; also in a quarter of a day?

(2) Divide $3417\frac{4}{9}$ by 45.

(3) If I spend alternately £$\frac{2}{7}$ and £$\frac{3}{7}$ in a day, how much shall I spend in 14 days, and how long will £$8\frac{4}{7}$ last me?

(4) Find the value of $\frac{11}{14}$ of a guinea + $\frac{3}{8}$ of a shilling — $\frac{5}{6}$ of 9*d.*

(5) Find the value of $\frac{13}{27}$ of £209. 11*s.* $2\frac{1}{4}$*d.*

(6) $2\frac{26}{35} \times 2, 3, 4, 5, 6, 7, 8, 9, 10, 35$.

(7) $2\frac{26}{35} \div 2, 3, 4, 5, 6, 7, 8, 9, 10$.

(8) $2\frac{26}{35} \div \frac{1}{35}, \frac{8}{35}, \frac{24}{35}, 1\frac{13}{35}$.

(9) If $\frac{5}{7}$ of a ship cost £3259. 11*s.* 8*d.*, what will the whole cost? Also what will $\frac{3}{7}$ cost?

(10) If a certain number of trusses of hay were accurately distributed among 359 horses, each would have $2\frac{82}{359}$ trusses. How many trusses were to be divided?

(11) Simplify $(9\frac{10}{11} + 11\frac{9}{11} + 13\frac{8}{11} + 15\frac{7}{11} + 17\frac{6}{11} + 19\frac{5}{11})$ — $(8\frac{7}{11} + 7\frac{5}{11} + 6\frac{9}{11} + 6\frac{2}{11} + 5\frac{6}{11} + 4\frac{10}{11})$.

(12) How many times is the difference between $\frac{1}{5}$ of $1\frac{7}{28}$ and $\frac{1}{7}$ of $1\frac{1}{4}$ contained in the sum of 3 times $10\frac{11}{28}$ and 5 times $9\frac{3}{140}$?

(13) Find the length of 7 pieces, if each is $12\frac{5}{9}$ yards long.

(14) Find the length of $5\frac{3}{4}$ pieces, if each is 60 yards long.

(15) Find the length of $147\frac{6}{7}$ pieces, if 9 pieces are $113\frac{2}{5}$ yards long.

CHAPTER II.

INTERCONVERSION OF DENOMINATORS.

§ 1. We have seen that by multiplying the numerator, the value of the fraction is multiplied; by multiplying the denominator, the value of the fraction is divided. Hence:

By multiplying both numerator and denominator *by the same number*, the value of the fraction is not altered. Thus $\frac{5}{7}=\frac{5\times6}{7\times6}=\frac{30}{42}$. We have, in fact, six times as many pieces as before, but each piece is one-sixth of the original size.

By dividing the numerator, the value of the fraction is divided; by dividing the denominator, the value of the fraction is multiplied. Hence:

By dividing both numerator and denominator by the same number, the value of the fraction is not altered. Thus: $\frac{30}{42}=\frac{30\div6}{42\div6}=\frac{5}{7}$. We have, in fact, one-sixth of the original number of pieces, but each piece is six times as great as before.

The numerator and denominator are called the *terms* of the fraction. By dividing numerator and denominator of a fraction by the same number, the fraction is reduced to lower terms, and when the terms are prime to each other, the fraction is at its *lowest terms.*

Reduce $\frac{61776}{80784}$ to lowest terms.

Applying the tests of Part I. Ch. XI. § 8, we instantly find that both terms are divisible by 4, $\therefore \frac{61776}{80784}=\frac{15444}{20196}$. This fraction is again reducible by 4, $\therefore \frac{15444}{20196}=\frac{3861}{5049}$. We have now got rid of all *even* common measures, and applying the test for 3 or 9, we find the fraction reducible by 9, $\therefore \frac{3861}{5049}=\frac{429}{561}$. Repeating this test, we find it reducible by 3 and not by 9, $\therefore \frac{429}{561}=\frac{143}{187}$. We have now got rid of all common measures which are multiples of 3. Applying the test for 11, we find the fraction reducible by 11, $\therefore \frac{143}{187}=\frac{13}{17}$. 13 and 17 being prime to each other, the fraction is reduced to its lowest terms.

Mod. op.: $4)\,\frac{61776}{80784} \Big| \; 4)\,\frac{15444}{20196} \Big| \; 9)\,\frac{3861}{5049} \Big| \; 3)\,\frac{429}{561} \Big| \; 11)\,\frac{143}{187} \Big| \; \frac{13}{17}$

Reduce $\frac{4189}{4307}$ to lowest terms. Here none of the tests above referred to reveal a common measure. Find G. C. M. of the terms. It is 59.

$$\begin{array}{r|l} \multicolumn{2}{c}{1,\ 71} \\ \hline 4189 & 4307 \\ 59 & \underline{118} \\ & 59 \end{array} \qquad \therefore \quad {}^{59)}\begin{array}{c|c} 4189 & 71 \\ \hline 4307 & 73 \end{array}$$

Reduce $\frac{1287}{4760}$ to lowest terms. The tests give us no common measure, and G. C. M. is found to be 1. Therefore the fraction is already at its lowest terms.

$$\begin{array}{c|c} 1287 & 4760 \\ \hline 117 & 119 \end{array}$$

Reduce $\frac{7850304}{13083840}$ to lowest terms.

$$\begin{array}{c|c|c|c|c|c|c} {}^{4)}\ 7850304 & {}^{4)}\ 1962576 & {}^{4)}\ 490644 & {}^{9)}\ 122661 & {}^{11)}\ 13629 & {}^{413)}\ 1239 & 3 \\ \hline 13083840 & 3270960 & 817740 & 204435 & 22715 & 2065 & 5 \end{array}$$

Here the first five divisors were found by inspection; the last by the process for G. C. M.

$$\begin{array}{c|c} 1239 & 2065 \\ \hline 413 & 413 \end{array}$$

To test the accuracy of the result, divide the original numerator by the last numerator, and the original denominator by the last denominator. The two quotients should obviously be the same.

$$\begin{array}{r} 3)\underline{7850304} \\ 2616768 \end{array} \qquad\qquad \begin{array}{r} 5)\underline{13083840} \\ 2616768 \end{array}$$

EXERCISE XIII.

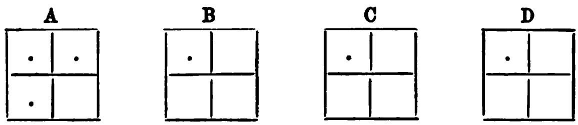
Reduce to lowest terms :

(1) $\frac{20}{25}$.	(14) $\frac{960}{1000}$.	(27) $\frac{44323}{61087}$.	(40) $\frac{6}{9}$.
(2) $\frac{36}{48}$.	(15) $\frac{35}{100}$.	(28) $\frac{339}{1243}$.	(41) $\frac{63}{99}$.
(3) $\frac{42}{77}$.	(16) $\frac{875}{10000}$.	(29) $\frac{1177}{2675}$.	(42) $\frac{74}{999}$.
(4) $\frac{18}{27}$.	(17) $\frac{4375}{10000}$.	(30) $\frac{11445}{15369}$.	(43) $\frac{27}{999}$.
(5) $\frac{28}{112}$.	(18) $\frac{224}{1000}$.	(31) $\frac{85359}{94128}$.	(44) $\frac{630}{9999}$.
(6) $\frac{56}{84}$.	(19) $\frac{2640}{2970}$.	(32) $\frac{85359}{86128}$.	(45) $\frac{143}{9999}$.
(7) $\frac{98}{112}$.	(20) $\frac{324}{1092}$.	(33) $\frac{6171}{6732}$.	(46) $\frac{9090}{9999}$.
(8) $\frac{75}{100}$.	(21) $\frac{924}{1092}$.	(34) $\frac{14141}{16289}$.	(47) $\frac{720}{99999}$.
(9) $\frac{1300}{1700}$.	(22) $\frac{6732}{9108}$.	(35) $\frac{881496}{104768}$.	(48) $\frac{8280}{99999}$.
(10) $\frac{625}{1200}$.	(23) $\frac{3872}{92807}$.	(36) $\frac{2760}{4485}$.	(49) $\frac{29810}{99999}$.
(11) $\frac{66}{121}$.	(24) $\frac{6640}{27360}$.	(37) $\frac{5760}{7000}$.	(50) $\frac{6216}{999999}$.
(12) $\frac{143}{176}$.	(25) $\frac{78473}{94653}$.	(38) $\frac{2205}{2240}$.	(51) $\frac{65065}{999999}$.
(13) $\frac{375}{1000}$.	(26) $\frac{17596}{26145}$.	(39) $\frac{14028}{28392}$.	(52) $\frac{8925345}{10881585}$.

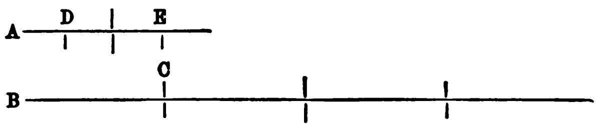
(53) If a cake is divided equally among a school of 75 pupils, 30 being boys, what part of the whole cake do the boys get, and what the girls?

(54) The property of a mining company is divided into 6000 shares. A holds 625 shares, B 800 shares, and C 900 shares. What part of the whole mine does each possess, and what part do they hold jointly?

§ 2. We have hitherto attached to the symbol $\frac{3}{4}$ the interpretation that *one* whole is divided into four equal parts, of which *three* are taken. If, however, *three* wholes are divided into four equal parts and *one* of these parts is taken, we obtain a piece of equal size, for a quarter of three must be three times as great as a quarter of one. Thus the same piece is obtained by taking three quarters of the square A, or one quarter of the three squares B, C and D.



Similarly, if the line A is one inch, and B three inches, one quarter of B, viz. BC, is three times as long as AD the quarter of A, and is therefore as long as AE.



Learn by heart: $\frac{1}{4}$ of $3 = \frac{3}{4}$ of 1. Thus:

$\frac{1}{4}$ of £3 = $\frac{1}{4}$ of 60*s.* = 15*s.*
$\frac{3}{4}$ of £1 = 3 × 5*s* = 15*s.*
$\frac{1}{4}$ of 3*s.* = $\frac{1}{4}$ of 36*d.* = 9*d.*
$\frac{3}{4}$ of 1*s.* = 3 × 3*d.* = 9*d.*

$\frac{3}{4}$ of £5. 11*s.* 8*d.* = $\frac{1}{4}$ of 3 × £5. 11*s.* 8*d.*

	£.	*s.*	*d.*
4)	5	11	8
	1	7	11
			3
	£4	3	9

	£.	*s.*	*d.*
	5	11	8
			3
4)	16	15	0
	£4	3	9

Similarly, $\frac{1}{7}$ of $5 = \frac{5}{7}$ of 1. Find $\frac{5}{7}$ of £8. 18*s*. $9\frac{1}{2}$*d*. in two ways.

7)8	18	$9\frac{1}{2}$		£8	18	$9\frac{1}{2}$
1	5	$6\frac{1}{2}$				5
		5		7)44	13	$11\frac{1}{2}$
£6	7	$8\frac{1}{2}$		£6	7	$8\frac{1}{2}$

Exercise XIV.

Find in two ways:

(1) $\frac{6}{7}$ of £940. 7*s*. 3*d*.

(2) $\frac{10}{11}$ of £1888. 13*s*. 1*d*.

(3) $\frac{5}{12}$ of £2060. 1*s*. 3*d*.

(4) $\frac{5}{6}$ of £21. 19*s*. 3*d*.

(5) $\frac{4}{11}$ of £25. 19*s*. $11\frac{3}{4}$*d*.

(6) $\frac{8}{9}$ of £17. 4*s*. $7\frac{1}{2}$*d*.

Exercise H.

(1) What is $\frac{1}{8}$ of 3? *Ans.* $\frac{3}{8}$.

(2) Divide 3 into 8 equal parts. *Ans.* $\frac{3}{8}$ to each part.

(3) If I spend 6*s*. in 7 days, what fraction of 1*s*. do I spend per day? *Ans.* $\frac{6}{7}$ of 1*s*.

(4) Distribute 7 yards into 11 equal parts. *Ans.* $\frac{7}{11}$ of a yard to each part.

(5) Divide 11 yards into 7 equal parts. *Ans.* $1\frac{4}{7}$ yards to each part.

(6) If 2 oranges are divided among 3 children, what will each child get? *Ans.* $\frac{2}{3}$ of an orange.

(7) Divide 4 cakes among 7 children. *Ans.* $\frac{4}{7}$ of a cake to each.

(8) Divide 7 cakes among 4 children. *Ans.* $\frac{7}{4}$ or $1\frac{3}{4}$ cakes to each.

(9) $\frac{1}{6}$ of 9. *Ans.* $1\frac{1}{2}$.

(10) $\frac{1}{9}$ of 6. " $\frac{6}{9}$ or $\frac{2}{3}$.

(11) $\frac{1}{8}$ of 15. " $1\frac{7}{8}$.

(12) $\frac{1}{15}$ of 8. " $\frac{8}{15}$.

(13) $\frac{1}{20}$ of 30. " $1\frac{1}{2}$.

(14) $\frac{1}{30}$ of 20. " $\frac{2}{3}$.

(15) $\frac{1}{30}$ of 25. " $\frac{5}{6}$.

(16) $\frac{1}{25}$ of 30. *Ans.* $1\frac{1}{5}$.

(17) $8 \div 9$. " $\frac{8}{9}$.

(18) $7 \div 10$. " $\frac{7}{10}$.

(19) $10 \div 7$. " $1\frac{3}{7}$.

(20) $15 \div 4$. " $\frac{15}{4}$ or $3\frac{3}{4}$.

(21) $67 \div 12$. " $5\frac{7}{12}$.

(22) $12 \div 67$. " $\frac{12}{67}$.

From this it follows that every fraction indicates a division, and every division a fraction, the dividend being the numerator, the divisor the denominator, and the quotient the value of the fraction. It is well to remember that

Dividend,	Divisor,	Quotient,
Numerator,	Denominator,	Fraction,

are two sets of names with identical meanings.

EXERCISE XV.

(1) $\frac{1}{74}$ of 111.
(2) $\frac{1}{111}$ of 74.
(3) Divide 60 things amongst 42 persons.
(4) „ 42 things amongst 60 persons.
(5) „ 108 things amongst 144 persons.
(6) „ 144 things into 108 equal parts.
(7) „ 520 things among 195 persons.
(8) „ 195 yards into 520 equal lengths.

§ 3. Distribute 75419 into 156 equal parts.

156) 75419 (483 to each part.
1301
539
71 over.

We can now dispose of this remainder 71. It is to be distributed into 156 equal parts, each part will therefore be $\frac{1}{156}$ of $71 = \frac{71}{156}$. The complete answer, therefore, is $483\frac{71}{156}$ to each part.

$15372 \div 792$.

792) 15372 ($19\frac{9}{22}$
7452
$\frac{324}{792} \Big| \frac{81}{198} \Big| \frac{9}{22}$ *Ans.* $19\frac{9}{22}$ to each part.

$16043 \div 72$.

8) 16043
9) $2005\frac{3}{8}$
$222\frac{59}{72}$. *Ans.* $222\frac{59}{72}$ to each part.

N.B. On comparing this process with that given in Part I. Ch. IX. § 15, et seq., it will be seen that they are identical.

EXERCISE XVI.

(1) $17429 \div 387$
(2) $150768 \div 1224$
(3) $150768 \div 132$
(4) $111114 \div 41$
(5) $86354 \div 45$
(6) $1000000 \div 1625$
(7) $195 \div 610$
(8) $26813 \div 73$
(9) $26813 \div 28$
(10) $3465 \div 5355$
(11) $155554 \div 2439$
(12) $73043 \div 42$
(13) $4016093 \div 1517$
(14) $18467 \div 2400$
(15) $77900 \div 1685$
(16) $12219 \div 165$
(17) $488 \div 2100$
(18) $2100 \div 488$.

§ 4. £517. 6*s*. 11*d*. ÷ 7.

7) 517 6 11
£73 18 $1\frac{4}{7}$

When there are no farthings in the dividend, the pence over should not be reduced to farthings, but at once expressed as a fraction of a penny.

Find in two ways $\frac{5}{9}$ of £417. 6*s.* 5*d.*

$$\begin{array}{r|rrr} 9) & 417 & 6 & 5 \\ \hline & 46 & 7 & 4\frac{5}{9} \\ & & & 5 \\ \hline & £231 & 16 & 10\frac{7}{9} \end{array} \qquad \begin{array}{rrrr} & 417 & 6 & 5 \\ & & & 5 \\ \hline 9) & 2086 & 12 & 1 \\ \hline & £231 & 16 & 10\frac{7}{9} \end{array}$$

£53. 8*s.* $9\frac{1}{4}d. \div 7$.

$$\begin{array}{r|rrr} 7) & 53 & 8 & 9\frac{1}{4} \\ \hline & £7 & 12 & 8\frac{5}{28} \end{array}$$

Here the $1\frac{1}{4}d.$ over must be made into farthings, i.e. fourths, $\frac{5}{4} \div 7 = \frac{5}{28}$.

£103. 8*s.* $10\frac{2}{5}d. \div 6$.

$$\begin{array}{r|rrr} 6) & 103 & 8 & 10\frac{2}{5} \\ \hline & £17 & 4 & 9\frac{22}{30}\Big|\frac{11}{15} \end{array}$$

Here the $4\frac{2}{5}$ over must be made into fifths of a penny. $\frac{22}{5} \div 6 = \frac{22}{30} = \frac{11}{15}$ of a penny.

£23281. 15*s.* $7\frac{1}{2}d. \div 615$.

$$\begin{array}{l} 615)23281 \;\; 15 \;\;\; 7\tfrac{1}{2}(37 \;\; 17 \;\;\; 1\tfrac{47}{82} \\ \qquad\quad 4831 \\ \qquad\quad\;\; 526 \\ \qquad\quad 1053 \\ \qquad\quad\;\; 4385 \\ \qquad\qquad\; 80 \\ \qquad\quad\;\; 967 \\ \qquad\quad\;\; 352 \\ \qquad\quad\;\; \frac{705}{2} \div 615 = \frac{705}{1230}\Big|\frac{141}{246}\Big|\frac{47}{82} \end{array}$$

Exercise XVII.

(1) £4617. 13*s.* 5*d.* ÷ 2, 3, 4, 5, 6, 7, 8, 9, 10.

(2) £4617. 13*s.* 5*d.* ÷ 17, 151, 367, 5928.

(3) £4617. 13*s.* 5*d.* ÷ 42, 63, 84, 108.

(4) £3815. 6*s.* $2\frac{3}{4}d.$ ÷ 2, 3, 4, 5, 6, 7, 8, 9, 10.

(5) £3815. 6*s.* $2\frac{3}{4}d.$ ÷ 17, 151, 367, 5928.

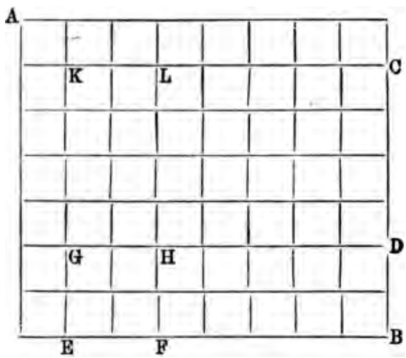
(6) £3815. 6*s.* $2\frac{3}{4}d.$ ÷ 42, 63, 84, 108.

(7) Find in two ways: $\frac{5}{9}$ of £23. 14*s.* 2*d.*, £23. 14*s.* $2\frac{1}{2}d.$, £23. 14*s.* $2\frac{3}{4}d.$, £23. 14*s.* $2\frac{5}{7}d.$

(8) Find $\frac{6}{11}$ of £897. 12*s.* 10*d.*, £897. 12*s.* $10\frac{1}{4}$*d.*, £897. 12*s.* $10\frac{3}{5}$*d.*
(9) Divide $3417\frac{4}{9}$ by 42 in three different ways.
(10) Find the value of £$\frac{6}{13}$, $\frac{7}{8}$ of a guinea, $\frac{3}{5}$ of 1*s.*

§ 5. COMPLEX FRACTIONS.

It is required to find $\frac{5}{7}$ of $\frac{3}{8}$. $\frac{1}{7}$ of $\frac{1}{8}=\frac{1}{56}$ (Part II. Ch. I. § 9), $\therefore$ $\frac{1}{7}$ of $\frac{3}{8}=\frac{3}{56}$, and $\therefore$ $\frac{5}{7}$ of $\frac{3}{8}=5\times\frac{3}{56}=\frac{15}{56}$. Similarly, $\frac{3}{8}$ of $\frac{5}{7}=3\times\frac{1}{8}$ of $\frac{5}{7}=3\times\frac{5}{56}=\frac{15}{56}$. Hence $\frac{5}{7}$ of $\frac{3}{8}=\frac{3}{8}$ of $\frac{5}{7}=\frac{15}{56}$.



From A to C is $\frac{1}{7}$ of the whole AB, $\therefore$ AD is $\frac{5}{7}$ of AB.

AG is $\frac{1}{8}$ of AD or $\frac{1}{8}$ of $\frac{5}{7}$, $\therefore$ AH is $\frac{3}{8}$ of $\frac{5}{7}$, or 15 times the piece AK, i.e. $\frac{15}{56}$ of AB.

Again, AE is $\frac{1}{8}$ of AB, $\therefore$ AF is $\frac{3}{8}$ of AB.

AL is $\frac{1}{7}$ of AF or $\frac{1}{7}$ of $\frac{3}{8}$, $\therefore$ AH is $\frac{5}{7}$ of $\frac{3}{8}=\frac{15}{56}$ as before.

Learn by heart: *To simplify a complex fraction, multiply the two numerators together for the new numerator, and the two denominators for the new denominator.*

EXERCISE K.

(1) $\frac{1}{4}$ of 3.	*Ans.* $\frac{3}{4}$.	(8) $\frac{1}{3}\times 4$.	*Ans.* $1\frac{1}{3}$.
(2) $\frac{1}{3}$ of 4.	„ $1\frac{1}{3}$.	(9) $\frac{1}{4}\times 3$.	„ $\frac{3}{4}$.
(3) $\frac{1}{3}$ of $\frac{1}{4}$.	„ $\frac{1}{12}$.	(10) $\frac{1}{4}\times 2$.	„ $\frac{1}{2}$.
(4) $\frac{1}{3}$ of $\frac{3}{4}$.	„ $\frac{1}{4}$.	(11) $\frac{3}{4}\times 2$.	„ $1\frac{1}{2}$.
(5) $\frac{1}{4}$ of $\frac{1}{3}$.	„ $\frac{1}{12}$.	(12) $\frac{3}{4}\times 3$.	„ $2\frac{1}{4}$.
(6) $\frac{1}{4}$ of $1\frac{1}{3}$.	„ $\frac{1}{3}$.	(13) $3\div 4$.	„ $\frac{3}{4}$.
(7) $\frac{1}{4}$ of $1\frac{1}{5}$.	„ $\frac{3}{10}$.	(14) $4\div 3$.	„ $1\frac{1}{3}$.

(15) $\frac{1}{3} \div 4$. *Ans.* $\frac{1}{12}$.
(16) $\frac{1}{4} \div 3$. ,, $\frac{1}{12}$.
(17) $\frac{3}{4} \div 3$. ,, $\frac{1}{4}$.
(18) $1\frac{1}{3} \div 4$. ,, $\frac{1}{3}$.
(19) $1\frac{2}{3} \div 4$. ,, $\frac{5}{12}$.
(20) $\frac{1}{7}$ of 6. ,, $\frac{6}{7}$.
(21) $\frac{1}{6}$ of 7. ,, $1\frac{1}{6}$.
(22) $\frac{1}{6}$ of $\frac{1}{7}$. ,, $\frac{1}{42}$.
(23) $\frac{1}{7}$ of $\frac{1}{6}$. ,, $\frac{1}{42}$.
(24) $\frac{1}{7}$ of $\frac{5}{6}$. ,, $\frac{5}{42}$.
(25) $\frac{1}{7}$ of $5\frac{3}{5}$. ,, $\frac{4}{5}$.
(26) $\frac{1}{7}$ of $5\frac{2}{5}$. ,, $\frac{27}{35}$.
(27) $5\frac{3}{5} \div 7$. ,, $\frac{4}{5}$.
(28) $5\frac{2}{5} \div 7$. ,, $\frac{27}{35}$.
(29) $5\frac{3}{5} \div \frac{2}{5}$. ,, 14.
(30) $5\frac{3}{5} \div 1\frac{2}{5}$. ,, 4.
(31) $6 \div 7$. ,, $\frac{6}{7}$.
(32) $7 \div 6$. ,, $1\frac{1}{6}$.
(33) $\frac{1}{7} \div 6$. ,, $\frac{1}{42}$.
(34) $\frac{1}{6} \div 7$. ,, $\frac{1}{42}$.
(35) $\frac{1}{7} \times 6$. ,, $\frac{6}{7}$.
(36) $\frac{1}{6} \times 7$. ,, $1\frac{1}{6}$.
(37) $\frac{1}{11}$ of 13. ,, $1\frac{2}{11}$.
(38) $\frac{1}{13}$ of $\frac{1}{11}$. ,, $\frac{1}{143}$.
(39) $\frac{1}{13}$ of 11. ,, $\frac{11}{13}$.
(40) $\frac{1}{11}$ of $\frac{1}{13}$. ,, $\frac{1}{143}$.
(41) $\frac{1}{13}$ of $\frac{4}{11}$. ,, $\frac{4}{143}$.
(42) $\frac{7}{13}$ of $\frac{4}{11}$. ,, $\frac{28}{143}$.
(43) $11 \div 13$. ,, $\frac{11}{13}$.
(44) $13 \div 11$. ,, $1\frac{2}{11}$.
(45) $\frac{1}{13} \div 11$. ,, $\frac{1}{143}$.
(46) $\frac{7}{13} \div 11$. ,, $\frac{7}{143}$.
(47) $\frac{11}{13} \div 11$. ,, $\frac{1}{13}$.
(48) $\frac{11}{13} \div \frac{1}{13}$. ,, 11.
(49) $\frac{1}{5}$ of 8. ,, $1\frac{3}{5}$.
(50) $\frac{1}{8}$ of 5. ,, $\frac{5}{8}$.

(51) $\frac{1}{8}$ of $\frac{1}{5}$. *Ans.* $\frac{1}{40}$.
(52) $\frac{1}{5}$ of $\frac{1}{8}$. ,, $\frac{1}{40}$.
(53) $\frac{1}{5}$ of $\frac{5}{8}$. ,, $\frac{1}{8}$.
(54) $\frac{1}{5}$ of $\frac{3}{8}$. ,, $\frac{3}{40}$.
(55) $\frac{3}{5}$ of $\frac{3}{8}$. ,, $\frac{9}{40}$.
(56) $\frac{4}{5}$ of $\frac{5}{8}$. ,, $\frac{1}{2}$.
(57) $\frac{1}{2}$ of $\frac{3}{4}$. ,, $\frac{3}{8}$.
(58) $\frac{5}{6}$ of $\frac{3}{7}$. ,, $\frac{5}{14}$.
(59) $\frac{2}{3}$ of $\frac{4}{5}$. ,, $\frac{8}{15}$.
(60) $\frac{1}{12}$ of 10. ,, $\frac{5}{6}$.
(61) $\frac{1}{10}$ of 12. ,, $1\frac{1}{5}$.
(62) $\frac{1}{12}$ of $\frac{1}{10}$. ,, $\frac{1}{120}$.
(63) $\frac{1}{10}$ of $\frac{1}{12}$. ,, $\frac{1}{120}$.
(64) $\frac{1}{12}$ of $\frac{7}{10}$. ,, $\frac{7}{120}$.
(65) $\frac{1}{10}$ of $\frac{7}{12}$. ,, $\frac{7}{120}$.
(66) $\frac{1}{10}$ of $2\frac{11}{12}$. ,, $\frac{7}{24}$.
(67) $\frac{1}{12}$ of $4\frac{4}{5}$. ,, $\frac{2}{5}$.
(68) $\frac{5}{12}$ of $4\frac{4}{5}$. ,, 2.
(69) $\frac{7}{12}$ of $4\frac{4}{5}$. ,, $2\frac{4}{5}$.
(70) $\frac{1}{12} \div 10$. ,, $\frac{1}{120}$.
(71) $\frac{1}{10} \div 12$. ,, $\frac{1}{120}$.
(72) $10 \div 12$. ,, $\frac{5}{6}$.
(73) $12 \div 10$. ,, $1\frac{1}{5}$.
(74) $4\frac{4}{5} \div 12$. ,, $\frac{2}{5}$.
(75) $4\frac{4}{5} \div 8$. ,, $\frac{3}{5}$.
(76) $4\frac{4}{5} \div \frac{3}{5}$. ,, 8.
(77) $4\frac{4}{5} \div 7$. ,, $\frac{24}{35}$.
(78) $4\frac{4}{5} \div 1\frac{3}{5}$. ,, 3.
(79) $4\frac{4}{5} \div \frac{2}{5}$. ,, 12.
(80) $4\frac{4}{5} \div 2$. ,, $2\frac{2}{5}$.
(81) $6 \div 9$. ,, $\frac{2}{3}$.
(82) $9 \div 6$. ,, $1\frac{1}{2}$.
(83) $\frac{1}{9}$ of 6. ,, $\frac{2}{3}$.
(84) $\frac{1}{6}$ of 9. ,, $1\frac{1}{2}$.
(85) $\frac{1}{6}$ of $\frac{1}{9}$. ,, $\frac{1}{54}$.
(86) $\frac{1}{6}$ of $\frac{5}{9}$. ,, $\frac{5}{54}$.

(87) $\frac{1}{6}$ of $\frac{5}{9}$ of 27. *Ans.* $2\frac{1}{2}$.
(88) $\frac{1}{6}$ of $\frac{5}{9}$ of 45. „ $4\frac{1}{6}$.
(89) $\frac{1}{4}$ of $\frac{5}{9}$. „ $\frac{5}{36}$.
(90) $\frac{5}{9}$ of $\frac{1}{4}$. „ $\frac{5}{36}$.
(91) $\frac{5}{9}$ of 4. „ $2\frac{2}{9}$.
(92) $\frac{1}{6}$ of $\frac{5}{9}$ of 4. „ $\frac{10}{27}$.
(93) $\frac{5}{8}$ of 20. „ $12\frac{1}{2}$.
(94) $\frac{1}{7}$ of $\frac{5}{8}$ of 20. „ $1\frac{11}{14}$.
(95) $\frac{3}{7}$ of $\frac{5}{8}$ of 20. „ $5\frac{5}{14}$.
(96) $\frac{4}{5}$ of $\frac{7}{11}$. „ $\frac{28}{55}$.
(97) $\frac{1}{2}$ of $\frac{4}{5}$ of $\frac{7}{11}$. „ $\frac{14}{55}$.
(98) $\frac{3}{5}$ of $\frac{7}{11}$. „ $\frac{21}{55}$.
(99) $\frac{1}{2}$ of $\frac{3}{5}$ of $\frac{7}{11}$. „ $\frac{21}{110}$.
(100) $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{7}{11}$. „ $\frac{14}{55}$.
(101) $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{11}$. *Ans.* $\frac{56}{165}$.
(102) $\frac{4}{5}$ of $\frac{7}{11}$ of $\frac{2}{3}$. „ $\frac{56}{165}$.
(103) $\frac{4}{5}$ of $\frac{2}{3}$ of $\frac{7}{11}$. „ $\frac{56}{165}$.
(104) $\frac{7}{11}$ of $\frac{2}{3}$ of $\frac{4}{5}$. „ $\frac{56}{165}$.
(105) $\frac{1}{15}$ of $\frac{1}{12}$. „ $\frac{1}{180}$.
(106) $\frac{1}{12}$ of $\frac{1}{15}$. „ $\frac{1}{180}$.
(107) $\frac{1}{15}$ of 12. „ $\frac{4}{5}$.
(108) $\frac{1}{12}$ of 15. „ $1\frac{1}{4}$.
(109) $\frac{1}{23}$ of 46. „ 2.
(110) $\frac{1}{46}$ of 23. „ $\frac{1}{2}$.
(111) $\frac{1}{60}$ of 320. „ $5\frac{1}{3}$.
(112) $\frac{1}{320}$ of 60. „ $\frac{3}{16}$.
(113) $\frac{4}{5}$ of $\frac{3}{11}$. „ $\frac{12}{55}$.
(114) $\frac{3}{8}$ of $\frac{7}{10}$. „ $\frac{21}{80}$.

EXERCISE XVIII.

Prove by finding values that:

(1) $\frac{5}{7}$ of £$\frac{3}{8}$ = $\frac{3}{8}$ of £$\frac{5}{7}$.
(2) $\frac{5}{7}$ of $\frac{3}{8}$ of 1 cwt. = $\frac{3}{8}$ of $\frac{5}{7}$ of 1 cwt.
(3) $\frac{5}{7}$ of $\frac{3}{8}$ of 1 yard = $\frac{3}{8}$ of $\frac{5}{7}$ of 1 yard.
(4) $\frac{2}{3}$ of $\frac{7}{11}$ of 1 mile = $\frac{7}{11}$ of $\frac{2}{3}$ of 1 mile.

§ 6. Simplify $\frac{4}{9}$ of $\frac{8}{11}$ of $\frac{12}{13}$ of 20.

$$\tfrac{4}{9} \text{ of } \tfrac{8}{11} = \frac{4\times 8}{9\times 11}$$

$$\tfrac{4}{9} \text{ of } \tfrac{8}{11} \text{ of } \tfrac{12}{13} = \frac{4\times 8\times 12}{9\times 11\times 13}$$

$$\tfrac{4}{9} \text{ of } \tfrac{8}{11} \text{ of } \tfrac{12}{13} \text{ of } 20 = \frac{4\times 8\times 12}{9\times 11\times 13} \text{ of } 20.$$

Since $\frac{1}{4}$ of 3 = $\frac{3}{4}$ of 1 (Part II. Ch. II. § 2), $\frac{1}{9\times 11\times 13}$ of 20 = $\frac{20}{9\times 11\times 13}$, and $\frac{4\times 8\times 12}{9\times 11\times 13}$ of $20 = \frac{4\times 8\times 12\times 20}{9\times 11\times 13} = \frac{7680}{1287} = 5\frac{1245}{1287} = 5\frac{415}{429}$.

Learn by heart: *To simplify a complex fraction, multiply together all the numerators for a new numerator, and all the denominators for a new denominator.*

Simplify $\frac{4}{21}$ of $\frac{9}{20}$ of $\frac{10}{11}$ of $\frac{15}{16}$ of $\frac{14}{25}$ of 11.

$$\tfrac{4}{21} \text{ of } \tfrac{9}{20} \text{ of } \tfrac{10}{11} \text{ of } \tfrac{15}{16} \text{ of } \tfrac{14}{25} \text{ of } 11 = \frac{4\times 9\times 10\times 15\times 14\times 11}{21\times 20\times 11\times 16\times 25}$$

$$400)\,\frac{831600}{1848000} \;\Big|\; 3)\,\frac{2079}{4620} \;\Big|\; 11)\,\frac{693}{1540} \;\Big|\; 7)\,\frac{63}{140} \;\Big|\; \frac{9}{20}$$

This reduction to lowest terms might have been performed before actually multiplying the several numerators and denominators.

Since 4×9 is 3 times as much as 4×3, $4\times 9\times 10\times 15\times 14\times 11$ is 3 times as much as $4\times 3\times 15\times 14\times 11$. Hence, dividing by 3 the one factor 9, divides the whole numerator by 3; and generally:—*A product is divided by any number if* ONE *of its factors is divided by that number*, and therefore any one factor of the numerator may be divided by a number, provided some one factor of the denominator is also divided by that number.

Note that this rule applies only to a series of factors, and not to a series of *addenda*, where *each* number must be divided in order to divide the *sum*.

The several numerators and denominators may therefore be "cancelled" against each other, thus:

4 and 16	become respectively	1 and 4
9 and 21	„	3 and 7
15 and 25	„	3 and 5
10 and 20	„	1 and 2
11 and 11	„	1 and 1

and the fraction will be

$$\frac{\overset{1}{\cancel{4}}}{\underset{7}{\cancel{21}}} \text{ of } \frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{20}}} \text{ of } \frac{\overset{1}{\cancel{10}}}{\underset{1}{\cancel{11}}} \text{ of } \frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{16}}} \text{ of } \frac{14}{\underset{5}{\cancel{25}}} \text{ of } \overset{1}{\cancel{11}} = \frac{1\times 3\times 1\times 3\times 14\times 1}{7\times 2\times 1\times 4\times 5}$$

which admits of further cancelling. 7 and 14 become respectively 1 and 2; and this 2 cancels against the 2 in the denominator, leaving $\frac{1\times 3\times 1\times 3\times 1\times 1}{1\times 1\times 1\times 4\times 5} = \frac{9}{20}$ as before.

Mod. op.:

$$\frac{\overset{1}{\cancel{4}}}{\underset{1}{\underset{\cancel{7}}{\cancel{21}}}} \text{ of } \frac{\overset{3}{\cancel{9}}}{\underset{1}{\underset{\cancel{2}}{\cancel{20}}}} \text{ of } \frac{\overset{1}{\cancel{10}}}{\underset{1}{\cancel{11}}} \text{ of } \frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{16}}} \text{ of } \frac{\overset{1}{\overset{\cancel{2}}{\cancel{14}}}}{\underset{5}{\cancel{25}}} \text{ of } \overset{1}{\cancel{11}} = \frac{9}{20}$$

Simplify $\frac{5}{8}$ of $\frac{3}{7}$ of $\frac{16}{25}$ of $\frac{4}{9}$ of $\frac{21}{55}$ of $6\frac{5}{12}$.

$$\frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{8}}} \text{ of } \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{7}}} \text{ of } \frac{\overset{2}{\cancel{16}}}{\underset{5}{\cancel{25}}} \text{ of } \frac{\overset{1}{\cancel{4}}}{\underset{1}{\underset{\cancel{3}}{\cancel{9}}}} \text{ of } \frac{\overset{1}{\overset{\cancel{3}}{\cancel{21}}}}{\underset{5}{\cancel{55}}} \text{ of } \frac{\overset{7}{\cancel{77}}}{\underset{3}{\cancel{12}}} = \frac{14}{75}$$

Exercise XIX.

(1) $\frac{2}{3}$ of $\frac{7}{11}$.

(2) $\frac{2}{3}$ of $\frac{5}{6}$.

(3) $\frac{3}{7}$ of $2\frac{1}{10}$.

(4) $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of 4.

(5) $\frac{3}{5}$ of $\frac{10}{27}$ of $\frac{9}{20}$ of $8\frac{1}{3}$.

(6) $\frac{8}{11}$ of $\frac{20}{21}$ of $\frac{35}{48}$ of $2\frac{10}{19}$.

(7) $\frac{8}{9}$ of $\frac{36}{37}$ of $\frac{5}{13}$ of 111.

(8) $\frac{7}{12}$ of $\frac{28}{29}$ of $\frac{26}{35}$ of $7\frac{1}{4}$.

(9) $\frac{2}{3}$ of $\frac{2}{3}$ of $\frac{2}{3}$ of $\frac{2}{3}$.

(10) $\frac{7}{25}$ of $3\frac{4}{7}$.

(11) $\frac{42}{43}$ of $\frac{13}{105}$ of $1\frac{7}{208}$.

(12) $\frac{5}{8}$ of $\frac{120}{121}$ of $\frac{66}{85}$ of 17.

(13) $\frac{14}{15}$ of $\frac{23}{35}$.

(14) $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{1}{4}$ of $\frac{1}{5}$ of $\frac{1}{6}$.

(15) $\frac{38}{39}$ of $\frac{52}{57}$ of $\frac{69}{86}$ of $1\frac{20}{23}$.

(16) $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$ of $\frac{8}{9}$ of $\frac{9}{10}$ of 10.

(17) $\frac{1}{10}$ of $\frac{1}{10}$ of $\frac{1}{10}$ of $\frac{1}{10}$ of $\frac{1}{10}$.

(18) $\frac{1}{10}$ of $\frac{2}{9}$ of $\frac{3}{8}$ of $\frac{4}{7}$ of $\frac{5}{6}$ of $1\frac{1}{5}$.

(19) $\frac{5}{11}$ of $\frac{34}{35}$ of $1\frac{4}{51}$.

(20) $\frac{7}{25}$ of $\frac{8}{11}$ of 30.

(21) $\frac{3}{5}$ of $\frac{6}{7}$ of $\frac{3}{11}$ of 4.

(22) $\frac{2}{3}$ of $\frac{4}{9}$ of $\frac{8}{27}$.

(23) $\frac{2}{21}$ of $\frac{7}{9}$ of $\frac{5}{8}$ of $\frac{4}{15}$ of 12.

(24) $\frac{8}{15}$ of $\frac{2}{3}$ of $\frac{12}{13}$ of $\frac{5}{9}$ of 39.

(25) $\frac{8}{15}$ of $\frac{9}{16}$ of $\frac{10}{33}$ of $\frac{11}{21}$ of $\frac{12}{35}$ of $\frac{13}{15}$ of $\frac{14}{39}$ of 20.

(26) $\frac{1}{7}$ of $20 + \frac{3}{7}$ of 61.

(27) $\frac{2}{11}$ of $7 - \frac{1}{22}$ of 6.

(28) $\frac{3}{13}$ of $2 + \frac{3}{13}$ of $3 + \frac{3}{13}$ of $4 + \frac{3}{13}$ of $5 + \frac{3}{13}$ of 6.

(29) $\frac{2}{5}$ of $2\frac{1}{2} + \frac{3}{7}$ of $2\frac{1}{3} + \frac{5}{9}$ of $1\frac{4}{5} + \frac{7}{26}$ of $3\frac{5}{7}$.

(30) $\frac{5}{12}$ of $16 + \frac{11}{12}$ of $20 + \frac{7}{12}$ of $36 + \frac{1}{12}$ of 8.

(31) $\frac{113}{355}$ of $\frac{85}{226}$ of $\frac{12}{35}$ of $1\frac{3}{4}$.

(32) $\frac{7}{8}$ of $\frac{3}{4}$ of $\frac{8}{21}$ of $\frac{4}{5}$ of $\frac{5}{8}$ of $\frac{3}{4}$ of 8.

(33) $\frac{2}{13}$ of $\frac{39}{40}$ of $\frac{52}{117}$.

(34) $\frac{9}{11}$ of $\frac{7}{12}$ of $\frac{22}{63}$ of 48.

(35) $\frac{28}{59}$ of $\frac{3}{8}$ of $\frac{59}{63}$.

(36) $\frac{39}{40}$ of $\frac{7}{65}$ of $\frac{25}{42}$ of 12.

CHAPTER III.

DIFFERENT DENOMINATORS.

§ 1. Add $\frac{1}{2}$ and $\frac{1}{3}$. "In Addition and Subtraction we must always have the *same kind* of units" (Part I. Ch. V. § 17); hence $\frac{1}{2}$ and $\frac{1}{3}$ cannot be added in their present shape.

Add $\frac{1}{2}$ and $\frac{1}{4}$. *Ans.* $\frac{3}{4}$, because $\frac{1}{2} = \frac{2}{4}$. Can we not similarly reduce $\frac{1}{2}$ into thirds? A whole has 3 thirds, therefore $\frac{1}{2} = \frac{1}{3} + \frac{1}{2}$ of $\frac{1}{3} = \frac{1}{3} + \frac{1}{6}$, therefore $\frac{1}{2} + \frac{1}{3} = (\frac{1}{3} + \frac{1}{6}) + \frac{1}{3} = \frac{2}{3} + \frac{1}{6}$. But as a whole is 6 sixths, $\frac{1}{3} = \frac{2}{6}$ and $\frac{2}{3} = \frac{4}{6}$; $\therefore \frac{1}{2} + \frac{1}{3} = \frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$.* The result, however, can be obtained by a simpler process.

$$1 = \tfrac{2}{2} = \tfrac{3}{3} = \tfrac{4}{4} = \tfrac{5}{5} = \tfrac{6}{6} = \tfrac{7}{7} = \tfrac{8}{8} = \tfrac{9}{9} = \tfrac{10}{10} = \tfrac{11}{11} = \tfrac{12}{12} = \tfrac{13}{13} = \tfrac{14}{14} = \tfrac{15}{15} = \tfrac{16}{16} = \tfrac{17}{17} = \tfrac{18}{18},$$

$$\therefore \tfrac{1}{2} = \tfrac{2}{4} = \tfrac{3}{6} = \tfrac{4}{8} = \tfrac{5}{10} = \tfrac{6}{12} = \tfrac{7}{14} = \tfrac{8}{16} = \tfrac{9}{18},$$

$$\tfrac{1}{3} = \tfrac{2}{6} = \tfrac{3}{9} = \tfrac{4}{12} = \tfrac{5}{15} = \tfrac{6}{18}.$$

We see from this that halves and thirds can both be converted into sixths, twelfths, eighteenths, &c.,

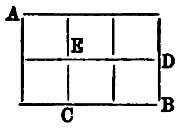
$$\therefore \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{3}{6} + \tfrac{2}{6} = \tfrac{5}{6},$$

$$\text{or} \quad = \tfrac{6}{12} + \tfrac{4}{12} = \tfrac{10}{12} = \tfrac{5}{6},$$

$$\text{or} \quad = \tfrac{9}{18} + \tfrac{6}{18} = \tfrac{15}{18} = \tfrac{5}{6},$$

and so on with other common multiples of 2 and 3. But the L.C.M. is obviously the best. Illustrations:

a)



$AB = 1$
$AC = \frac{1}{3}$
$AD = \frac{1}{2}$
$AE = \frac{1}{6}$

$$\therefore AC = \tfrac{2}{6}, \quad AD = \tfrac{3}{6}; \quad \therefore \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6}.$$

b) $\frac{1}{2}$ of 1*s.* = 6*d.*, $\frac{1}{3}$ of 1*s.* = 4*d.* $(\frac{1}{2} + \frac{1}{3})$ of 1*s.* = 6*d.* + 4*d.* = 10*d.* = $\frac{5}{6}$ of 1*s.*

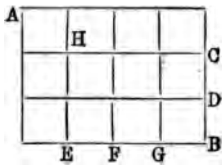
* This mode of reasoning is here given historically, as being that most commonly suggested by the more intelligent pupils.

Exercise L.

To be worked mentally from diagrams drawn on the black board.

(1) $\frac{1}{2}+\frac{1}{3}$. *Ans.* $\frac{5}{6}$.
(2) $\frac{1}{2}-\frac{1}{3}$. „ $\frac{1}{6}$.
(3) $\frac{1}{2}\div\frac{1}{6}$. „ 3 times.
(4) $\frac{1}{3}\div\frac{1}{6}$. „ Twice.
(5) $\frac{1}{6}\times 3$. „ $\frac{3}{6}$ or $\frac{1}{2}$.
(6) $\frac{1}{6}\times 2$. *Ans.* $\frac{2}{6}$ or $\frac{1}{3}$.
(7) $\frac{2}{3}\div\frac{1}{6}$. „ 4 times.
(8) $(\frac{2}{3}+\frac{1}{2})\div\frac{1}{6}$. 7 times.
(9) $\frac{2}{3}+\frac{1}{2}$. „ $\frac{7}{6}=1\frac{1}{6}$.
(10) $\frac{2}{3}-\frac{1}{2}$. „ $\frac{1}{6}$.

(11) A post is driven through water into the mud below; $\frac{1}{2}$ is buried in the mud, $\frac{1}{3}$ is under water, and there are 5 feet above water. Find the length of the post. *Ans.* 30 feet.



(12) $\frac{1}{3}+\frac{1}{4}$. *Ans.* $\frac{7}{12}$.
(13) $\frac{1}{3}-\frac{1}{4}$. „ $\frac{1}{12}$.
(14) $\frac{3}{4}+\frac{2}{3}$. „ $1\frac{5}{12}$.
(15) $\frac{3}{4}-\frac{2}{3}$. „ $\frac{1}{12}$.
(16) $\frac{1}{3}$ of $\frac{1}{4}$. „ $\frac{1}{12}$.
(17) $\frac{1}{4}$ of $\frac{1}{3}$. *Ans.* $\frac{1}{12}$.
(18) $\frac{2}{3}\div\frac{1}{12}$. „ 8 times.
(19) $\frac{3}{4}\div\frac{1}{12}$. „ 9 times.
(20) $1\frac{3}{4}+1\frac{1}{3}$. „ $3\frac{1}{12}$.
(21) $1\frac{3}{4}-1\frac{2}{3}$. „ $\frac{1}{12}$.

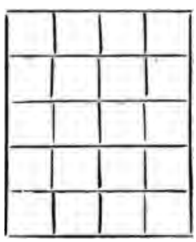
Reduce to lowest terms:

(22) $\frac{2}{12}$. *Ans.* $\frac{1}{6}$. (24) $\frac{4}{12}$. *Ans.* $\frac{1}{3}$. (26) $\frac{8}{12}$. *Ans.* $\frac{2}{3}$. (28) $\frac{10}{12}$. *Ans.* $\frac{5}{6}$.
(23) $\frac{3}{12}$. „ $\frac{1}{4}$. (25) $\frac{6}{12}$. „ $\frac{1}{2}$. (27) $\frac{9}{12}$. „ $\frac{3}{4}$.

(29) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}$. *Ans.* $1\frac{3}{12}|\frac{1}{4}$.
(30) $1\frac{1}{2}+2\frac{2}{3}+3\frac{1}{4}+2\frac{5}{6}$. *Ans.* $10\frac{3}{12}|\frac{1}{4}$.

(31) If $(\frac{1}{4}+\frac{1}{3})$ of a sum of money is £3. 10*s.*, what is the whole sum? *Ans.* £6.

(32) On Monday I spent $\frac{2}{3}$ of my money, on Tuesday $\frac{1}{4}$ of it, and had then £1. 10*s.* left. How much had I at first? *Ans.* £18.



(33) $\frac{1}{4}+\frac{1}{5}$. *Ans.* $\frac{9}{20}$.
(34) $\frac{1}{4}-\frac{1}{5}$. " $\frac{1}{20}$.
(35) $\frac{1}{4}$ of $\frac{1}{5}$. " $\frac{1}{20}$.
(36) $\frac{1}{5}$ of $\frac{1}{4}$. " $\frac{1}{20}$.
(37) $\frac{3}{4}+\frac{2}{5}$. " $1\frac{3}{20}$.
(38) $\frac{3}{4}-\frac{2}{5}$. *Ans.* $\frac{7}{20}$.
(39) $\frac{3}{4}$ of $\frac{2}{5}$. " $\frac{3}{10}$.
(40) $\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{10}+\frac{1}{20}$. " $1\frac{1}{10}$.
(41) $2\frac{1}{2}+1\frac{3}{4}+3\frac{2}{5}$. " $7\frac{13}{20}$.
(42) Express $\frac{2}{5}$ as a fraction whose denominator is 20. *Ans.* $\frac{8}{20}$.
(43) " $\frac{3}{10}$ " " " $\frac{6}{20}$.
(44) " $\frac{1}{2}$ " " " $\frac{10}{20}$.
(45) " $\frac{3}{4}$ " " " $\frac{15}{20}$.
(46) " $\frac{7}{10}$ " " " $\frac{14}{20}$.
(47) $2\frac{1}{5}\div\frac{1}{20}$. *Ans.* 44 times.
(48) $2\frac{1}{4}-1\frac{2}{5}$. *Ans.* $\frac{13}{20}$.
(49) $13\frac{1}{2}-(1\frac{3}{4}+2\frac{2}{5}+1\frac{7}{10})$. " $7\frac{13}{20}$.
(50) $(\frac{1}{4}+\frac{1}{5})+(\frac{1}{4}-\frac{1}{5})+(\frac{1}{4}$ of $\frac{1}{5})$. " $\frac{11}{20}$.

Reduce to lowest terms :

(51) $\frac{18}{20}$. *Ans.* $\frac{9}{10}$. (53) $\frac{15}{20}$. *Ans.* $\frac{3}{4}$. (55) $\frac{10}{20}$. *Ans.* $\frac{1}{2}$. (57) $\frac{4}{20}$. *Ans.* $\frac{1}{5}$.
(52) $\frac{16}{20}$. " $\frac{4}{5}$. (54) $\frac{12}{20}$. " $\frac{3}{5}$. (56) $\frac{5}{20}$. " $\frac{1}{4}$. (58) $\frac{8}{20}$. " $\frac{2}{5}$.

Exercise XX.

Shew by finding values that :

(1) $(\frac{1}{2}+\frac{1}{3})$ of £1 $=\frac{5}{6}$ of £1.
(2) " 13*s.* 9*d.* $=\frac{5}{6}$ of 13*s.* 9*d.*
(3) " 1 oz. troy $=\frac{5}{6}$ of 1 oz. troy.
(4) " 1 yard $=\frac{5}{6}$ of 1 yard.
(5) $(\frac{3}{4}+\frac{2}{3})$ of £1 $=1\frac{5}{12}$ of £1.
(6) " 13*s.* 9*d.* $=1\frac{5}{12}$ of 13*s.* 9*d.*
(7) " 1 oz. troy $=1\frac{5}{12}$ of 1 oz. troy.
(8) " 1 yard $=1\frac{5}{12}$ of 1 yard.
(9) " £2. 11*s.* 6*d.* $=1\frac{5}{12}$ of £2. 11*s.* 6*d.*
(10) $(\frac{3}{4}-\frac{2}{5})$ of £1 $=\frac{7}{20}$ of £1.
(11) $(\frac{4}{5}+\frac{1}{4})$ of 1 oz. troy $=1\frac{1}{20}$ oz. troy.

§ 2. $\frac{5}{8}+\frac{7}{12}$. L. C. M. of 8 and 12 is 24.

$$1=\tfrac{24}{24};\ \tfrac{1}{8}=\tfrac{3}{24};\ \tfrac{5}{8}=\tfrac{15}{24}$$
$$\tfrac{1}{12}=\tfrac{2}{24};\ \tfrac{7}{12}=\tfrac{14}{24}$$
$$\therefore \tfrac{5}{8}+\tfrac{7}{12}=\frac{15+14}{24}=\tfrac{29}{24}=1\tfrac{5}{24}$$

Mod. op.:

$$\begin{array}{r|l} \multicolumn{2}{c}{24} \\ 3 & 15 \\ 2 & 14 \\ \hline \multicolumn{2}{r}{\frac{29}{24}=1\frac{5}{24}} \end{array}$$

Wording: 8 in 24, 3′; 3 × 5 = 15′; 12 in 24, 2′; 7 × 2 = 14′, &c.

$\frac{1}{2}+\frac{2}{3}+\frac{4}{9}+\frac{7}{12}+\frac{5}{6}$. L.C.M. of 2, 3, 9, 12, 6, is 36.

$$\begin{array}{r|l} \multicolumn{2}{c}{36} \\ - & 18 \\ 12 & 24 \\ 4 & 16 \\ 3 & 21 \\ 6 & 30 \\ \hline \multicolumn{2}{c}{36)\,109\,(3\frac{1}{36}} \\ \multicolumn{2}{c}{1} \end{array}$$

$8\frac{13}{48}+19\frac{31}{45}+429\frac{17}{18}+\frac{17}{60}$. L.C.M. of 48, 45, 18, 60 = 720.

$$\begin{array}{r} 8 \\ 19 \\ 429 \\ \hline 456 \end{array} \qquad \begin{array}{r|l} \multicolumn{2}{c}{720} \\ 15 & 195 \\ 16 & 496 \\ 40 & 680 \\ 12 & 204 \\ \hline \end{array}$$

$$\overset{9)}{\frac{1575}{720}} \Bigg| \overset{5)}{\frac{175}{80}} \Bigg| \frac{35}{16} = 2\frac{3}{16}. \qquad \textit{Ans. } 458\frac{3}{16}.$$

$$\frac{1}{2} \text{ of } \frac{4}{5}+\frac{2}{3} \text{ of } \frac{7}{8}+\frac{1}{12} \text{ of } 8+\frac{3}{8}\div 5+\frac{7}{15}\times 4+\frac{49}{16}$$
$$= \frac{2}{5} + \frac{7}{12} + \frac{2}{3} + \frac{3}{40} + 1\frac{13}{15} + 3\frac{1}{16}.$$

L.C.M. of 5, 12, 3, 40, 15, 16 = 240.

$$\begin{array}{r} \\ \\ \\ \\ 1 \\ 3 \\ \hline 4 \end{array} \qquad \begin{array}{r|r} \multicolumn{2}{c}{240} \\ 48 & 96 \\ 20 & 140 \\ 80 & 160 \\ 6 & 18 \\ 16 & 208 \\ - & 15 \\ \hline \multicolumn{2}{c}{240)\,637\,(2} \\ \multicolumn{2}{c}{157} \end{array}$$

Ans. $6\frac{157}{240}$.

$$9\frac{5}{6}-7\frac{3}{8}=9\frac{20}{24}-7\frac{9}{24}=2\frac{11}{24}$$
$$7\frac{5}{21}-2\frac{11}{14}=7\frac{10}{42}-2\frac{33}{42}=4\frac{19}{42}$$

Exercise XXI.

(1) $\frac{4}{5}+\frac{5}{6}$; $\frac{3}{4}+\frac{7}{8}$; $\frac{1}{2}+\frac{1}{6}$; $\frac{1}{2}-\frac{1}{6}$; $\frac{6}{7}-\frac{4}{9}$; $\frac{4}{15}+\frac{11}{20}$; $\frac{5}{16}+\frac{11}{24}$.

(2) $12\frac{5}{8}+7\frac{3}{16}$; $12\frac{5}{8}-7\frac{3}{16}$; $85\frac{7}{12}+27\frac{11}{18}$; $85\frac{7}{12}-27\frac{11}{18}$.

(3) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$; $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$; $\frac{5}{6}+\frac{11}{12}+\frac{8}{15}+\frac{7}{20}+\frac{13}{30}$.

(4) $5\frac{17}{20}+11\frac{19}{30}+24\frac{21}{40}+\frac{9}{50}+17\frac{8}{15}+14+11\frac{5}{12}$.

(5) $9\frac{4}{7}+15\frac{11}{28}+103\frac{17}{63}+1\frac{11}{42}+10\frac{1}{4}$.

(6) $1473-279$; $1473\frac{5}{13}-279$; $1473-279\frac{11}{12}$; $1473\frac{5}{13}-279\frac{11}{12}$; $1473\frac{7}{18}-279\frac{11}{12}$.

(7) $\frac{5}{14}+7\frac{8}{25}+11\frac{9}{16}+10\frac{11}{30}+14\frac{5}{8}+100+77\frac{6}{35}$.

(8) $\frac{5}{14}+\frac{6}{11}+9\frac{1}{2}$; $20\frac{5}{12}+11\frac{7}{20}+5\frac{1}{8}+305$; $278\frac{15}{16}-30\frac{5}{12}$.

(9) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}$; $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}+\frac{7}{8}+\frac{8}{9}+\frac{9}{10}$.

(10) $\frac{1}{2}-\frac{1}{3}$; $\frac{1}{3}-\frac{1}{4}$; $\frac{1}{4}-\frac{1}{5}$; $\frac{1}{5}-\frac{1}{6}$; $\frac{1}{6}-\frac{1}{7}$; $\frac{1}{7}-\frac{1}{8}$; $\frac{1}{8}-\frac{1}{9}$; $\frac{1}{9}-\frac{1}{10}$.

(11) $\frac{5}{17}+\frac{11}{34}+\frac{14}{51}+\frac{19}{68}$; $\frac{11}{38}+\frac{14}{57}+\frac{17}{76}$; $\frac{9}{13}+\frac{5}{39}+\frac{17}{52}$.

(12) $118\frac{5}{11}-17\frac{3}{14}$; $94\frac{5}{11}-91\frac{13}{14}$; $125\frac{5}{22}-10\frac{17}{33}$; $40\frac{1}{2}-30\frac{47}{90}$.

(13) $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}$; $\frac{4}{7}+\frac{2}{9}-\frac{3}{5}+\frac{11}{21}$.

(14) $\frac{1}{2}$ of $\frac{1}{3}+\frac{1}{3}$ of $\frac{1}{4}$; $\frac{2}{5}$ of $\frac{10}{11}+\frac{3}{8}$ of $\frac{16}{33}$; $\frac{1}{12}$ of $11+\frac{1}{11}$ of 12.

(15) $\frac{2}{11}$ of $210+\frac{7}{12}$ of $210+\frac{5}{8}$ of $7\frac{1}{11}$.

(16) $3\frac{5}{24}+7\frac{11}{12}+8\frac{13}{16}+9\frac{14}{15}$.

(17) $\frac{1}{7}$ of $5\frac{4}{9}+\frac{2}{13}$ of $1\frac{2}{37}+\frac{2}{3}$ of $\frac{17}{37}$.

(18) $\frac{7}{20}$ of $63-\frac{3}{50}$ of $7\frac{1}{2}$.

(19) $\frac{15}{16}$ of $7\frac{3}{5}+\frac{8}{17}$ of $10\frac{1}{5}-\frac{5}{7}$ of $2\frac{9}{10}$.

(20) *a.* $\frac{3}{10}+\frac{7}{100}+\frac{9}{1000}+\frac{5}{10000}$; $\frac{8}{100}+\frac{17}{10000}+\frac{9}{1000000}$.

b. $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\frac{1}{100000}$.

c. $\frac{1}{10}+\frac{2}{100}+\frac{3}{1000}+\frac{4}{10000}+\frac{5}{100000}$.

d. $\frac{1}{10}+\frac{1}{1000}+\frac{1}{100000}+\frac{1}{10000000}$.

e. $\frac{7}{10}+\frac{9}{100}+\frac{4}{10000}$.

f. $\frac{3}{100}+\frac{4}{1000}+\frac{7}{1000000}$.

g. $\frac{17}{10}+\frac{17}{100}+\frac{17}{1000}+\frac{17}{10000}+\frac{17}{100000}$.

h. $\frac{143}{100}+\frac{2471}{1000}+\frac{82643}{1000000}$.

(21) Which is the greater of each of the following pairs, and by how much : $\frac{5}{8}$ or $\frac{7}{11}$; $\frac{4}{17}$ or $\frac{15}{67}$; $\frac{4}{17}$ or $\frac{15}{68}$; $\frac{2}{11}$ or $\frac{9}{50}$?

(22) Also of $\frac{1}{2}$ of $\frac{1}{3}$ and $\frac{1}{4}$ of $\frac{3}{5}$; $\frac{2}{11}$ of $3\frac{1}{2}$ and $\frac{4}{11}$ of $1\frac{3}{4}$?

(23) Add together the sum and the difference of $\frac{9}{20}$ of $7\frac{3}{8}$, and $\frac{5}{14}$ of $3\frac{3}{40}$.

(24) Find the difference between $\frac{1}{3}$ and $\frac{3}{10}$; $\frac{4}{11}$ and $\frac{36}{100}$; $\frac{6}{7}$ and $\frac{857142}{1000000}$; $\frac{1}{960}$ and $\frac{1}{1000}$.

(25) $43\frac{7}{15} - 1\frac{1}{3} - 1\frac{31}{48} - 1\frac{23}{24} - 2\frac{13}{48} - 2\frac{7}{12} - 2\frac{43}{48} - 3\frac{5}{12}$.

(26) $43\frac{7}{15} - (1\frac{1}{3} + 1\frac{31}{48} + 1\frac{23}{24} + 2\frac{13}{48} + 2\frac{7}{12} + 2\frac{43}{48} + 3\frac{5}{12})$.

(27) $(\frac{1}{2} + \frac{4}{13} + 7\frac{9}{40} + 8\frac{14}{39} + 7\frac{1}{4} + 8\frac{3}{10} + 4\frac{1}{12}) - 36\frac{1}{40}$.

(28) $(8\frac{5}{18} + 9\frac{10}{27} + 17\frac{11}{36} + 40) - (30\frac{13}{40} + 11\frac{11}{20})$.

(29) $(172\frac{19}{78} + 93\frac{14}{117}) + (172\frac{19}{78} - 93\frac{14}{117})$.

(30) $(172\frac{19}{78} + 93\frac{14}{117}) - (172\frac{19}{78} - 93\frac{14}{117})$.

(31) $6\frac{3}{4} + \frac{27}{5} + \frac{5}{12} \times 3 + \frac{7}{18} \times 5 + 6\frac{2}{5} \div 4 + 1\frac{1}{8} \div 2 + \frac{5}{9}$ of $\frac{3}{4}$.

(32) A bequeathed to his two sons $\frac{1}{4}$ of his property each, to each of his three daughters $\frac{1}{9}$ of his property, to his nephew $\frac{1}{18}$ of his property, and a like sum to his niece; the remainder, £1000, to a hospital. Find the value of the whole property, and the shares in money of the several heirs.

(33) A was condemned to pay $\frac{11}{96}$ of the costs of a law-suit; B had to pay $\frac{11}{39}$ of the same costs; C had to pay the remainder, amounting to £12. 11*s.* Find the costs of the whole suit.

(34) In a cricket-match, 11 players made a certain number of runs; the first made $\frac{1}{10}$ of the total number, the next three each $\frac{5}{36}$, the next five each $\frac{1}{15}$, and the two last 18 runs between them. The other side made successively $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{12}$, $\frac{1}{15}$, $\frac{1}{20}$, $\frac{1}{30}$, $\frac{1}{40}$ and $\frac{1}{60}$ of their opponents' total. Which side had won, and by how much?

§ 3. Multiplication by Fractions.

£7. 13*s.* 6*d.* × $\frac{2}{3}$. We have shewn (Part I. Ch. V. § 17) that in multiplication one of the factors must be so many *times*. £7. 13*s.* 6*d.* cannot mean *times*; what sense, then, are we to attach to "$\frac{2}{3}$ times"? £7. 13*s.* 6*d.* × 5 may be interpreted: Find the cost of 5 articles at £7. 13*s.* 6*d.* each; and similarly, £7. 13*s.* 6*d.* × $\frac{2}{3}$ would be interpreted: Find the cost of $\frac{2}{3}$ of an article at £7. 13*s.* 6*d.* for one.

This cost will evidently be $\frac{3}{8}$ *of* £7. 13*s.* 6*d.* Are we, then, to conclude generally that $\times$ means *of*? Certainly not, until we have tested this extension of meaning by the three principles laid down above (Part I. Ch. V. § 15).

(*a*) It *has* an intelligible meaning.

(*b*) It does "not alter the sense attached to the symbol in the earlier cases," for, calling £7. 13*s.* 6*d.* "a collection," multiplying that collection by 5 means taking 5 *of* them. Does it "remain subject to the general rules already established"? The only general rules hitherto established are that multiplication may be performed in any order; thus $5 \times 3 = 3 \times 5$; and that though one of the numbers must mean *times*, it is indifferent which does so; thus $£5 \times 3 = £3 \times 5$.

We have shewn above (Part II. Ch. II. § 5) that $\frac{5}{7}$ of $\frac{3}{8} = \frac{3}{8}$ of $\frac{5}{7}$, and it follows immediately that $\frac{5}{7}$ of $£\frac{3}{8} = \frac{3}{8}$ of $£\frac{5}{7}$, and we can also see that $\frac{3}{8}$ of £7. 13*s.* 6*d.* $= 7 \times £\frac{3}{8} + 13 \times \frac{3}{8}s. + 6 \times \frac{3}{8}d.$

Definition of multiplication: *Treat the multiplicand as a unit, and take as many* OF *that unit, or such a part* OF *that unit, as is indicated by the multiplier.*

Hence, generally: $\times$ = OF.

N.B. Beginners are apt to connect the word *of* with division, but division can only be interpreted by "*of*" by altering the divisor; thus to divide by 2, 3, &c., is to take $\frac{1}{2}$, $\frac{1}{3}$, &c., and not 2, 3, &c., *of* the dividend.

We now see that multiplication of fractions by fractions is equivalent to simplification of complex fractions.

§ 4. We might have anticipated this conclusion from our earlier notions of multiplication, viz. that half the multiplier yields half the product; a third of the multiplier yields a third of the product; two-thirds of the multiplier yields two-thirds of the product; and so on. Compare:

$$£12 \times 6 = £72.$$
$$£12 \times (\tfrac{1}{2} \text{ of } 6) = \tfrac{1}{2} \text{ of } £72 = £36.$$
$$£12 \times (\tfrac{1}{3} \text{ of } 6) = \tfrac{1}{3} \text{ of } £72 = £24.$$
$$£12 \times (\tfrac{2}{3} \text{ of } 6) = \tfrac{2}{3} \text{ of } £72 = £48.$$

The conclusion of § 3, however, shews that this holds true also for multipliers below unity; thus:

$$\begin{aligned}
&£12 \times 1 = £12. \\
&£12 \times (\tfrac{1}{2} \text{ of } 1) = \tfrac{1}{2} \text{ of } £12 = £6. \\
&£12 \times (\tfrac{1}{3} \text{ of } 1) = \tfrac{1}{3} \text{ of } £12 = £4. \\
&£12 \times (\tfrac{2}{3} \text{ of } 1) = \tfrac{2}{3} \text{ of } £12 = £8.
\end{aligned}$$

We have hitherto connected multiplication with increase, agreeing therein with the derivation of the word; but we have already seen that this connection fails in the case of multiplication by 1, where the multiplicand remains unaltered, and now we see that when the multiplier is *less* than 1, the product is *less* than the multiplicand.

$$\tfrac{2}{5} \times 1\tfrac{7}{8} \times 12 \times 5\tfrac{1}{4} \times \tfrac{6}{7} \; (= \tfrac{2}{5} \text{ of } \tfrac{15}{8} \text{ of } 12 \text{ of } \tfrac{21}{4} \text{ of } \tfrac{6}{7})$$

$$= \frac{\overset{1}{\cancel{2}} \times \overset{3}{\cancel{15}} \times \overset{3}{\cancel{12}} \times \overset{3}{\cancel{21}} \times \overset{3}{\cancel{6}}}{\underset{1}{\cancel{5}} \times \underset{\underset{2}{\cancel{4}}}{\cancel{8}} \times \underset{1}{\cancel{4}} \times \underset{1}{\cancel{7}}} = \frac{81}{2} = 40\tfrac{1}{2}.$$

The step in parentheses may be omitted.

$$\tfrac{16}{35} \times 1\tfrac{7}{8} \times \tfrac{21}{25} \times 3\tfrac{1}{3} \times 10 \times 5\tfrac{5}{8} \times \tfrac{1}{7}$$

$$= \frac{\overset{\overset{1}{\cancel{2}}}{\cancel{16}} \times \overset{\overset{1}{\cancel{5}}}{\cancel{15}} \times \overset{3}{\cancel{21}} \times \overset{\overset{1}{\cancel{2}}}{\cancel{10}} \times \overset{\overset{1}{\cancel{2}}}{\cancel{10}} \times 45 \times 1}{\underset{7}{\cancel{35}} \times \underset{1}{\cancel{8}} \times \underset{\underset{1}{\cancel{5}}}{\cancel{25}} \times \underset{1}{\cancel{3}} \times \underset{\underset{\underset{1}{\cancel{2}}}{\cancel{4}}}{\cancel{8}} \times \underset{1}{\cancel{7}}} = \frac{135}{7} = 19\tfrac{2}{7}.$$

EXERCISE XXII.

(1) $\frac{1}{2} \times \frac{1}{3}$; $\frac{1}{7} \times \frac{1}{8}$; $\frac{1}{20} \times \frac{3}{5}$; $\frac{15}{16} \times \frac{18}{25}$; $\frac{4}{21} \times \frac{28}{29}$.

(2) $4\frac{1}{2} \times 6$; $4\frac{1}{2} \times 6\frac{1}{2}$; $4\frac{1}{3} \times 6\frac{1}{4}$; $100\frac{3}{8} \times 4\frac{4}{11}$.

(3) $7\frac{1}{4} \times \frac{18}{25} \times \frac{13}{58} \times \frac{8}{39} \times 5$; $4\frac{1}{2} \times 5\frac{1}{4} \times 5\frac{1}{3} \times 13$.

(4) $\frac{1}{10} \times \frac{1}{10}$; $\frac{3}{10} \times \frac{7}{10}$; $\frac{11}{100} \times \frac{7}{10000}$; $\frac{13}{100} \times \frac{13}{10000} \times \frac{13}{1000000}$.

(5) $5\frac{7}{10} \times 81\frac{13}{100} \times 4\frac{7}{1000}$; $7\frac{1}{10} \times 8\frac{1}{10} \times 9\frac{1}{10}$.

(6) $2\frac{1}{4} \times 20\frac{1}{4} \times \frac{26}{27}$; $\frac{5}{38} \times \frac{57}{65} \times 3\frac{1}{4}$; $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$; $4\frac{1}{5} \times 4\frac{1}{5} \times 4\frac{1}{5} \times 4\frac{1}{5}$.

§ 5. Reciprocals.

$\frac{7}{8} \times \frac{8}{7} = 1$. If the product of two numbers is unity, each is called the RECIPROCAL of the other. Thus 1 is the reciprocal of 1, 3 of $\frac{1}{3}$, $\frac{1}{3}$ of 3, $\frac{2}{3}$ of $1\frac{1}{2}$, $1\frac{1}{2}$ of $\frac{2}{3}$, &c.

Exercise M.

(1) Find the reciprocal of $\frac{5}{8}$. . . . . *Ans.* $1\frac{3}{5}$.
(2) " $2\frac{1}{2}$. . . . . " $\frac{2}{5}$.
(3) " $6\frac{2}{3}$. . . . . " $\frac{3}{20}$.
(4) " $\frac{2}{9}$. . . . . " $4\frac{1}{2}$.
(5) " 7. . . . . " $\frac{1}{7}$.
(6) " $\frac{9}{10}$. . . . . " $1\frac{1}{9}$.
(7) By what number must $\frac{4}{11}$ be multiplied to give for product 1?
Ans. $2\frac{3}{4}$.
(8) By what must 5 be multiplied to give for product 1? " $\frac{1}{5}$.
(9) By what $6\frac{2}{3}$ be multiplied to give 1? " $\frac{3}{20}$.
(10) Given product 1, one factor $\frac{5}{8}$, find the other factor. " $1\frac{3}{5}$.
(11) By what must any number be multiplied to give 1?
Ans. By its reciprocal.
(12) By what must a number be multiplied to give 2?
Ans. By twice its reciprocal.
(13) By what must a number be multiplied to give 11?
Ans. By 11 times its reciprocal.
(14) By what must a number be multiplied to give $\frac{1}{2}$?
Ans. By $\frac{1}{2}$ of its reciprocal.
(15) By what must a number be multiplied to give $8\frac{2}{9}$?
Ans. By $8\frac{2}{9}$ times its reciprocal.
(16) Write out ten pairs of reciprocals.

§ 6. If 2 miles take 1 hour, 1 mile takes $\frac{1}{2}$ of an hour.

If	3	"	1	"	1	"	$\frac{1}{3}$	"
If	4	"	1	"	1	"	$\frac{1}{4}$	"
If	7	"	1	"	1	"	$\frac{1}{7}$	"
If	40	"	1	"	1	"	$\frac{1}{40}$	"

and so on.

Again:

If $\frac{1}{2}$	of a mile take	1	hour,	1	mile takes	2	hours.
If $\frac{1}{3}$	,,	1	,,	1	,,	3	,,
If $\frac{1}{4}$	,,	1	,,	1	,,	4	,,
If $\frac{1}{7}$	,,	1	,,	1	,,	7	,,
If $\frac{1}{40}$	,,	1	,,	1	,,	40	,,

and so on.

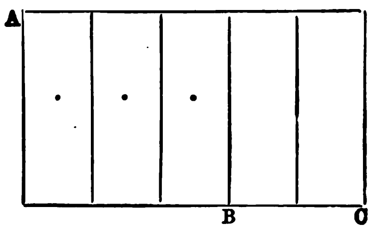
Again:

If $\frac{2}{9}$ of 1 mile take 1 hour, 1 mile will take $\frac{1}{2}$ of 9 hours $= \frac{9}{2}$ of 1 hour.

If $\frac{4}{11}$ of 1 mile take 1 hour, 1 mile will take $\frac{1}{4}$ of 11 hours $= \frac{11}{4}$ of 1 hour.

If $\frac{9}{10}$ of 1 mile take 1 hour, 1 mile will take $\frac{1}{9}$ of 10 hours $= \frac{10}{9}$ of 1 hour,
and so on.

Illustration: If AB is $\frac{3}{5}$ of AC, then AC is $\frac{5}{3}$ of AB.



Exercise N.

(1) If A has $\frac{3}{5}$ of B's money, what part of A's money has B?

Ans. $\frac{5}{3}$ of A's money.

(2) If to every gallon of milk we add $\frac{2}{9}$ of a gallon of water, how much milk must be added to a gallon of water?

Ans. $4\frac{1}{2}$ gallons of milk.

(3) A lb. avoirdupois is $1\frac{31}{144}$ of 1 lb. troy; what fraction of a lb. avoirdupois is 1 lb. troy? *Ans.* $\frac{144}{175}$ of 1 lb. av.

(4) A score is $1\frac{2}{3}$ dozen; what will a dozen cost if a score costs a doubloon?

Ans. $\frac{3}{5}$ of a doubloon.

(5) If $\frac{7}{8}$ of an article cost a guinea, what will the whole cost?

Ans. $\frac{8}{7}$ of a guinea = £1. 4s.

(6) If in working a question I obtain the result that 1 apple is to be given to $\frac{4}{5}$ of a boy, to what rational answer is this equivalent?

Ans. 1 boy gets $\frac{5}{4}$ of an apple.

§ 7. Division by Fractions.

$\frac{2}{3} \div \frac{9}{10}$. What does this mean? In Part I. Ch. VIII. §§ 1 and 5, we have stated that the sign $\div$ bears *two* interpretations, either of which we may adopt if intelligible. In this case, *neither* is intelligible; for, 1st, $\frac{9}{10}$ being larger than $\frac{2}{3}$ cannot be contained in it; and 2nd, we cannot attach any idea to a distribution of $\frac{2}{3}$ into $\frac{9}{10}$ equal parts. A further extension of the meaning of the symbol $\div$ becomes necessary.

$$\text{£}15 \div \text{£}3 = 5 \text{ times, because } 3 \times 5 = 15.$$
$$\text{£}15 \div 3 = \text{£}5, \quad \text{because } 3 \times 5 = 15.$$

Thus we see that, under either interpretation, divisor × quotient = dividend. It follows that division means in each case: *What must we multiply* $\left\{\begin{array}{l}\textit{the divisor by} \\ \textit{by the divisor}\end{array}\right\}$ *to get the dividend?* This meaning applies to the question proposed, which is therefore, By what number must $\frac{9}{10}$ be multiplied to give $\frac{2}{3}$? or, What must be multiplied by $\frac{9}{10}$ to give $\frac{2}{3}$?

If we multiply it (viz. $\frac{9}{10}$) by $\frac{10}{9}$, we obtain unity; but we wish to obtain, not unity, but only $\frac{2}{3}$ of unity; we ought therefore to have multiplied only by $\frac{2}{3}$ of $\frac{10}{9}$. Hence, $\frac{2}{3} \div \frac{9}{10} = \frac{2}{3}$ of $\frac{10}{9} = \frac{2}{3} \times \frac{10}{9}$. We thus see that to divide by $\frac{9}{10}$ is equivalent to multiplying by its reciprocal, $\frac{10}{9}$. But this process of reasoning holds for any other fractional divisor. Hence, generally: *To divide by a fraction, multiply by its reciprocal.* The same rule evidently holds also for integral divisors, for to divide any number by 3 is to take $\frac{1}{3}$ of it, i.e. to multiply it by $\frac{1}{3}$, and so on.

Learn by heart: *To divide by any number, multiply by its reciprocal.*

(*a*) $489 \div 7 = \frac{1}{7}$ of $489 = \frac{1}{7} \times 489 = \frac{489}{7} = 69\frac{6}{7}$.

(*b*) $\frac{4}{9} \div 7 = \frac{1}{7}$ of $\frac{4}{9} = \frac{4}{9} \times \frac{1}{7} = \frac{4}{63}$. (Cf. Part II. Ch. I. p. 14.)

(*c*) $\frac{3}{11} \div \frac{1}{11} = \frac{3}{11} \times 11 = 3$. (Cf. Part II. Ch. I. p. 16.)

(*d*) $\frac{7}{12} \div \frac{5}{8} = \frac{7}{\underset{3}{\cancel{12}}} \times \frac{\overset{2}{\cancel{8}}}{5} = \frac{14}{15}$. Verification: $\frac{\overset{7}{\cancel{14}}}{\underset{3}{\cancel{15}}} \times \frac{\overset{1}{\cancel{5}}}{\underset{4}{\cancel{8}}} = \frac{7}{12}$.

(*e*) $2\frac{5}{8} \div 2\frac{4}{5} = \frac{21}{8} \div \frac{14}{5} = \frac{\overset{3}{\cancel{21}}}{8} \times \frac{5}{\underset{2}{\cancel{14}}} = \frac{15}{16}$.

Verification: $\frac{15}{16} \times 2\frac{4}{5} = \frac{\overset{3}{\cancel{15}}}{\underset{8}{\cancel{16}}} \times \frac{\overset{7}{\cancel{14}}}{\underset{1}{\cancel{5}}} = \frac{21}{8} = 2\frac{5}{8}$.

We see now in its entirety that which indeed was dimly perceived before: (1st) that multiplication and division are opposite operations, by which we mean that they neutralize each other; (2nd) that they are interconvertible; every multiplication is a division by the reciprocal of the multiplier, and every division is a multiplication by the reciprocal of the divisor.

§ 8. We might have anticipated this conclusion from our earlier notions of division, viz. that half the divisor yields twice the quotient; a third of the divisor yields three times the quotient; twice the divisor yields half the quotient; therefore two-thirds ($\frac{2}{3}$) of the divisor yields half of three times the quotient, i.e. three halves ($\frac{3}{2}$) of the quotient.

$$\begin{aligned} &£12 \div 6 = £2. \\ &£12 \div (\tfrac{1}{2} \text{ of } 6) = 2 \times £2 = £4. \\ &£12 \div (\tfrac{1}{3} \text{ of } 6) = 3 \times £2 = £6. \\ &£12 \div (\tfrac{2}{3} \text{ of } 6) = \tfrac{3}{2} \times £2 = £3. \end{aligned}$$

The conclusion of § 7, however, shews that this holds true also of divisors below unity, thus:

$$\begin{aligned} &£12 \div 1 = £12. \\ &£12 \div (\tfrac{1}{2} \text{ of } 1) = 2 \times £12 = £24. \\ &£12 \div (\tfrac{1}{3} \text{ of } 1) = 3 \times £12 = £36. \\ &£12 \div (\tfrac{2}{3} \text{ of } 1) = \tfrac{3}{2} \times £12 = £18. \end{aligned}$$

We have hitherto connected division with decrease, but we see *that this connection* fails in the case of division by 1, which leaves *the dividend unaltered*, and now we see that when the divisor is *less than 1, the quotient* is *more* than the dividend.

Exercise XXIII.

(1) $8 \div \frac{2}{5}$.
(2) $\frac{2}{5} \div 8$.
(3) $\frac{4}{5} \div \frac{3}{7}$.
(4) $\frac{3}{7} \div \frac{4}{5}$.
(5) $1\frac{4}{5} \div \frac{1}{5}$.
(6) $1\frac{4}{5} \div 5$.
(7) $1\frac{4}{5} \div 3$.
(8) $1\frac{4}{5} \div \frac{3}{5}$.
(9) $1\frac{4}{5} \div 1\frac{1}{8}$.
(10) $\frac{1}{5} \div 1\frac{4}{5}$.
(11) $\frac{3}{5} \div 1\frac{4}{5}$.
(12) $3 \div 1\frac{4}{5}$.
(13) $8\frac{4}{7} \div 4\frac{4}{9}$.
(14) $11 \div 76$.
(15) $11 \div 55$.
(16) $55 \div 11$.
(17) $1\frac{5}{7} \div \frac{2}{3}$.
(18) $1\frac{5}{7} \div 1\frac{2}{3}$.
(19) $5\frac{5}{8} \div 15$.
(20) $5\frac{5}{8} \div 3\frac{3}{4}$.
(21) $10 \div 2\frac{1}{2}$.
(22) $25 \div 3\frac{1}{8}$.
(23) $\frac{15}{16} \div 7\frac{1}{2}$.
(24) $\frac{11}{12} \div 1\frac{3}{8}$.
(25) $\frac{19}{20} \div 7\frac{3}{5}$.
(26) $\frac{25}{26} \div 1\frac{1}{4}$.
(27) $\frac{13}{15} \div 6\frac{1}{9}$.
(28) $19\frac{1}{3} \div \frac{29}{48}$.
(29) $5\frac{5}{6} \div \frac{21}{22}$.
(30) $18\frac{3}{4} \div 1\frac{7}{8}$.
(31) $9\frac{3}{8} \div 4\frac{2}{7}$.
(32) $100\frac{5}{9} \div 8\frac{4}{9}$.
(33) $23\frac{1}{3} \div 11\frac{2}{3}$.
(34) $588\frac{1}{8} \div 91\frac{1}{8}$.
(35) $1819\frac{3}{10} \div 81\frac{5}{7}$.
(36) $21 \div (\frac{15}{19} \text{ of } 3\frac{4}{5})$.
(37) $18\frac{117}{256} \div (\frac{5}{8} \text{ of } 33\frac{3}{4})$.
(38) $3\frac{1}{3} \div (\frac{2}{3} \times 1\frac{1}{3})$.
(39) $(3\frac{1}{11} \times 5\frac{1}{17}) \div 1720$.
(40) $(3\frac{1}{2} \times \frac{8}{9} \text{ of } \frac{4}{7}) \div 1\frac{3}{5}$.
(41) $(\frac{28}{59} \times \frac{59}{63}) \div (\frac{1}{6} \times 4 \times \frac{2}{3})$.
(42) $(13\frac{5}{8} \times \frac{2}{3} \times \frac{9}{65}) \div (\frac{2}{3} \text{ of } 1\frac{44}{65} \times 1\frac{1}{8})$.
(43) £8. 7*s.* 10*d.* $\div 1\frac{2}{5}$.
(44) 13*s.* $8\frac{1}{4}$*d.* $\div 4\frac{7}{8}$.
(45) 4 days, 5 hours $\div 1\frac{5}{7}$.
(46) 3 qrs., 5 lbs., 8 oz. $\div 6\frac{2}{5}$.
(47) Find the cost of 1 article if $\frac{7}{10}$ cost 6*s.* 5*d.*
(48) If I earn 8*s.* 5*d.* in $1\frac{3}{5}$ days, how much is that a-day?
(49) If a soldier step $\frac{3}{4}$ of a yard, how many steps will he take in $1\frac{7}{8}$ miles?
(50) Divide $(\frac{17}{20} + \frac{11}{15} + \frac{7}{10} + \frac{4}{5})$ by $(\frac{17}{20} - \frac{11}{15} + \frac{7}{10} - \frac{4}{5})$.
(51) Divide $(4\frac{1}{7} - 2\frac{1}{4})$ by $(6\frac{1}{2} - 2\frac{1}{7})$.

Exercise XXIV.

(Miscellaneous Questions on the four Rules.)

(1) Find the sum of $4\frac{7}{12}$, $5\frac{2}{9}$, $7\frac{13}{16}$, and $10\frac{11}{20}$.
(2) What quantity exceeds $5\frac{2}{9}$ by $4\frac{7}{8}$?
(3) From what quantity must $6\frac{3}{5}$ be deducted to leave $\frac{1}{2}$ of $3\frac{1}{9}$?

(4) There are two fractions, the less is $10\frac{14}{25}$, their difference is $6\frac{8}{15}$. Find the greater.

(5) If from a certain quantity $2\frac{6}{7}$ be taken, $4\frac{11}{14}$ is left. Find the quantity.

(6) Of two weavers, A and B, A wove $9\frac{7}{10}$ pieces more than B, who wove $6\frac{11}{40}$ pieces. Find the total quantity woven.

(7) $5\frac{8}{11}$ exceeds a certain fraction by $(4 \div 2\frac{1}{3})$. Find the fraction.

(8) What fraction falls short of $\frac{7}{12}$ by $\frac{3}{20}$?

(9) What fraction is that to which $\frac{5}{76}$ must be added to give $\frac{11}{57}$?

(10) There are two fractions, the greater is $12\frac{7}{18}$, and their difference is $7\frac{53}{54}$. Find the less.

(11) What fraction increased by $\frac{1}{100}$ becomes $\frac{1}{10}$?

(12) Find a fraction which, repeated 3 times and increased by $14\frac{2}{9}$, makes 100.

(13) Find a fraction which, repeated 3 times and diminished by $14\frac{2}{9}$, makes 100.

(14) In a pair of scales, one contains $7\frac{4}{15}$ lbs., the other contains $11\frac{7}{9}$ lbs. Find the number of lbs. which drags down the heavier scale.

(15) Find the product of $4\frac{4}{9}$ and $3\frac{3}{5}$?

(16) What fraction must be divided by $7\frac{1}{2}$ to yield $7\frac{1}{2}$?

(17) From what number or fraction can $4\frac{17}{36}$ be taken 9 times exactly?

(18) From what fraction can $3\frac{5}{8}$ be taken $2\frac{1}{2}$ times, leaving remainder $3\frac{1}{2}$?

(19) Of what fraction is $10\frac{14}{15}$ the 10th part?

(20) What fraction divided by $4\frac{5}{7}$ gives the quotient $\frac{2}{3}$?

(21) What is the 7th part of $1\frac{1}{2}$?

(22) What fraction is that of which we must take $6\frac{2}{3}$ to get $5\frac{1}{2}$?

(23) By what fraction must 10 be multiplied to give 7?

(24) The product of two fractions is $\frac{5}{8}$; one factor is $1\frac{1}{4}$. Find the other.

(25) Given divisor $3\frac{1}{5}$, quotient $3\frac{1}{5}$. Find dividend.

(26) Given dividend $\frac{11}{12}$, quotient $6\frac{1}{2}$. Find divisor.

(27) Given dividend $\frac{11}{12}$, divisor $6\frac{1}{2}$. Find quotient.

(28) Given dividend $12\frac{17}{72}$, quotient 3, remainder $1\frac{5}{12}$. Find divisor.

§ 9. Literal Summary of the Rules on Fractions.

Let a, b, c, d, m, &c., represent any integers whatever.

Formulæ.	*Examples.*
(α) $\frac{a}{b} = (\frac{1}{b} \text{ of } 1) \times a = \frac{1}{b} \text{ of } a.$ (Ch. I. § 3, Ch. II. § 2.)	$\frac{2}{3} = (\frac{1}{3} \text{ of } 1) \times 2 = \frac{1}{3} \text{ of } 2$
(β)* If $a < b$, $\frac{a}{b} < 1.$	$\frac{7}{12} < 1$
If $a = b$, $\frac{a}{b} = 1.$	$\frac{12}{12} = 1$
† If $a > b$, $\frac{a}{b} > 1.$ (Ch. I. § 4.)	$\frac{13}{12} > 1$
(γ) $a + \frac{b}{c} = \frac{a \times c + b}{c}$ (Ch. I. § 5.)	$4\frac{5}{7} = \frac{4 \times 7 + 5}{7} = \frac{33}{7}$
(δ) $\frac{a}{b} = a \div b.$ (Ch. I. § 6, Ch. II. § 2.)	$\frac{23}{5} = 23 \div 5 = 4\frac{3}{5}$
(ϵ) $\frac{a}{d} + \frac{b}{d} + \frac{c}{d} = \frac{a+b+c}{d}$	$\frac{4}{19} + \frac{6}{19} + \frac{7}{19} = \frac{4+6+7}{19} = \frac{17}{19}$
$\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}$ (Ch. I. § 7.)	$\frac{17}{19} - \frac{12}{19} = \frac{17-12}{19} = \frac{5}{19}$
(ζ) $\frac{a}{b} \times m = \frac{a \times m}{b} = \frac{a}{b \div m}$ (Ch. I. § 8.)	$\frac{7}{18} \times 6 = \frac{7 \times 6}{18} = \frac{42}{18} = 2\frac{6}{18}$; or, $\frac{7}{18} \times 6 = \frac{7}{18 \div 6} = \frac{7}{3} = 2\frac{1}{3}$
(η) $\frac{a}{b} \div m = \frac{a \div m}{b} = \frac{a}{b \times m}$ (Ch. I. § 9.)	$\frac{12}{13} \div 4 = \frac{12 \div 4}{13} = \frac{3}{13}$; or $= \frac{12}{13 \times 4} = \frac{12}{52}$
(θ) $\frac{a}{b} = \frac{a \times m}{b \times m} = \frac{a \div m}{b \div m}$ (Ch. II. § 1.)	$\frac{20}{25} = \frac{20 \times 3}{25 \times 3} = \frac{60}{75}$; or $= \frac{20 \div 5}{25 \div 5} = \frac{4}{5}$
(ι) $\frac{1}{a} \text{ of } \frac{1}{b} = \frac{1}{a \times b}$ (Ch. II. § 5.)	$\frac{1}{4} \text{ of } \frac{1}{5} = \frac{1}{4 \times 5} = \frac{1}{20}$
(κ) $\frac{a}{b} \text{ of } \frac{c}{d} = \frac{a \times c}{b \times d}$ (Ch. II. § 6.)	$\frac{5}{7} \text{ of } \frac{3}{8} = \frac{5 \times 3}{7 \times 8} = \frac{15}{56}$

* Read, "If a is less than b."

† Read, "If a is greater than b."

Formulæ.	*Examples.*
(λ) * $\frac{a}{b} \pm \frac{c}{d} = \frac{a \times d \pm c \times b}{b \times d}$ (Ch. III. §§ 1, 2.)	$\frac{5}{8} \pm \frac{3}{7} = \frac{5 \times 7 \pm 3 \times 8}{7 \times 8} = \frac{35 \pm 24}{56} = \frac{59}{56}$ or $\frac{11}{56}$
(μ) $\frac{a}{b} \times \frac{c}{d} = \frac{a}{b}$ of $\frac{c}{d}$ (Ch. III. § 3.)	$\frac{2}{5} \times \frac{3}{7} = \frac{2}{5}$ of $\frac{3}{7}$
(ν) $\frac{a}{b} \times \frac{b}{a} = 1.$ (Ch. III. § 5.)	$\frac{4}{9} \times \frac{9}{4} = 1$
(ξ) $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ (Ch. III. § 7.)	$\frac{4}{11} \div \frac{5}{7} = \frac{4}{11} \times \frac{7}{5} = \frac{4 \times 7}{11 \times 5} = \frac{28}{55}$

CHAPTER IV.

THE UNITARY METHOD.

§ 1. Find the cost of 27 articles, if 12 articles cost £4. 17*s.* 6*d.*

Statement: 12 articles cost £4. 17*s.* 6*d.*
Question: 27 " £*x*,

where *x* stands for "the quantity to be found."

Mod. op.: 12 articles cost............ £4. 17*s.* 6*d.*
1 article costs ...$\frac{1}{12}$ of £4. 17*s.* 6*d.*
27 articles cost ...$\frac{27}{12}$ of £4. 17*s.* 6*d.*

$$\frac{\overset{9}{\cancel{27}}}{\underset{4}{\cancel{12}}} \text{ of } £4.\ 17s.\ 6d. = \frac{£43.\ 17s.\ 6d.}{4} = £10.\ 19s.\ 4\tfrac{1}{2}d.$$

Ans. £10. 19*s.* 4½*d.*

* Read, $\frac{a}{b}$ plus or minus $\frac{c}{d}$.

EXERCISE XXV. (*a*).

(1)	Find the cost of	30	articles, if	18	cost	£5. 7*s*. 6*d*.
(2)	"	45	"	20	"	£7. 17*s*. 6*d*.
(3)	"	60	"	25	"	£0. 17*s*. 10*d*.
(4)	"	150	"	210	"	£15. 15*s*.
(5)	"	210	"	150	"	£15. 15*s*.
(6)	"	68	"	153	"	£37. 8*s*. 9*d*.

(7) If it takes $7\frac{1}{2}$ hours to travel 150 miles, how long will it take to travel 220 miles?

(8) If 1800 men require 475 cwt. of food, how much will 2250 men require?

(9) If I pay £3. 15*s*. for the loan of £100, what shall I pay for the loan of £385?

(10) What shall I pay for the loan of £566 at 5 per cent. (i.e. at £5 for every £100)?

(11) Find the interest on £439 at $7\frac{1}{2}$ per cent.

(12) Find the interest on £1050 at $7\frac{3}{8}$ per cent.

§ 2. How many articles can be bought for £28, if 11 articles cost £15. 8*s*.?

Statement: £15. 8*s*. buy 11* articles.
Question: £28 " x "

Mod. op.: £15. 8*s*. = 308*s*. buy ....11 articles.
1*s*. buys......$\frac{1}{308}$ of 11 articles.
£28. 0*s*. = 560*s*. buy ......$\frac{560}{308}$ of 11 articles.

$$\frac{\overset{\overset{20}{\cancel{140}}}{\cancel{560}} \times \overset{1}{\cancel{11}}}{\underset{\underset{1}{\cancel{11}}}{\underset{\cancel{77}}{\cancel{308}}}} = 20.$$

Ans. 20 articles.

* Always place last the quantity of the same denomination as the answer required.

EXERCISE XXV. (*b*).

(1) If for £17. 18*s*. 4*d*. I can buy 60 articles, how many can I buy for £10. 15*s*.?

(2) If 30 men earn £53. 3*s*. $1\frac{1}{2}$*d*. in a week, how many men will it take to earn £177. 3*s*. 9*d*.?

(3) If 10 oz. of gold cost £38. 18*s*. 9*d*., how many ounces can be bought for £408. 16*s*. $10\frac{1}{2}$*d*.?

(4) If iron plates cost £20 per ton, what weight can I buy for 17*s*. 6*d*.?

(5) If £3. 15*s*. pays for the loan of £80, how much can I borrow for £5?

(6) How much money at 5 per cent. can I borrow with £72. 15*s*. a year?

(7) How much at £3. 15*s*. per cent. can I borrow with £100?

(8) If an agent is paid £2. 10*s*. for every £100 worth of goods sold, how much must he sell to earn £27. 15*s*.?

(9)	Find the interest on	£777. 15*s*.	at $7\frac{1}{2}$	per cent.
(10)	”	£843. 10*s*.	at 4	”
(11)	”	£1050.	at $3\frac{3}{4}$	”
(12)	”	£19.	at 10	”

§ 3. If 3 cwt., 3 qrs., 21 lbs. cost £3. 18*s*. 9*d*., what will 2 tons 17 cwt., 3 qrs., 10 lbs. cost?

Statement: 3 cwt., 3 qrs., 21 lbs. cost £3. 18. 9*d*.
Question: 2 tons, 17 cwt., 3 qrs., 10 lbs. cost £x.

Mod. op.:

3 cwt., 3 qrs., 21 lbs. = 441 lbs. cost .... £3. 18*s*. 9*d*.
1 lb. costs ... $\frac{1}{441}$ of £3. 18*s*. 9*d*.
2 tons, 17 cwt., 3 qrs., 10 lbs. = 6478 lbs. cost ... $\frac{6478}{441}$ of £3. 18*s*. 9*d*.

$\frac{6478}{441}$ of £3. 18*s*. 9*d*. = £57. 16*s*. $9\frac{3}{7}$*d*.

Ans. £57. 16*s*. $9\frac{3}{7}$*d*.

EXERCISE XXV. (c).

(1) If 15 tons cost £17. 10*s.*, what will 8 tons, 15 cwt. cost?

(2) If 6 cwt., 1 qr., 15 lbs. cost £13. 14*s.* 7½*d.*, what will 18 cwt., 19 lbs. cost?

(3) If 5 oz., 10 dwts. of gold cost £21. 8*s.* 3¾*d.*, what is the value of 3 oz., 5 dwts.?

(4) If 1 lb. troy of silver is worth £3. 3*s.*, what is the value of 6 silver spoons each weighing 4 oz., 11 dwts.?

(5) If on a velocipede I can travel 20½ miles in 54 minutes, how long would it take me to travel 35 miles?

(6) If the sun passes over 360° of longitude in 24 hours, how long will it take from the meridian of London to that of New York, which are 75° apart?

(7) If for the loan of £325. 10*s.* I pay £16. 5*s.* 6*d.*, what should I pay for the loan of £59. 15*s.*?

(8) If a partner holding £747. 12*s.* 6*d.* in a concern draws a profit of £56. 1*s.* 5¼*d.*, what ought a partner holding £812. 15*s.* to draw?

(9) If for the loan of a sum of money for 7 months I pay £7. 17*s.* 6*d.*, what ought I to pay for a year?

(10) If I pay £1. 17*s.* 10*d.* commission on an amount of £75. 13*s.* 4*d.*, how much is that per cent. (i.e. on £100)?

(11) If a house rated at £45 pays a tax of £3. 11*s.* 8*d.*, what must a house be rated at that pays a tax of £5. 14*s.* 8*d.*?

(12) If Government Stock costing £92. 12*s.* 6*d.* yields a profit of £3. 10*s.*, what profit should I get from Stock costing £271. 14*s.*?

(13) 3000 Prussian feet exceed 3000 French feet by 100 Prussian feet; how many French feet are there in 1000 Prussian feet?

(14) If 8 kilometres are 4 miles, 7 fur., 168 yds., how many miles are there in 11 kilometres, and how many kilometres are there in 11 miles?

(15) If 7 doz. and 9 bottles of wine cost £22. 1*s.* 9*d.*, how much wine shall I get for £11. 17*s.* 6*d.*?

§ 4. If $\frac{5}{8}$ of an article cost £1. 17*s.* 6*d.*, what would $1\frac{7}{12}$ articles cost?

Statement: $\frac{5}{8}$ articles cost £1. 17*s.* 6*d.*
Question: $1\frac{7}{12}$ " £*x*.

Mod. op.: $\frac{5}{8}=\frac{15}{24}$ articles cost ....£1. 17*s.* 6*d.*
$\frac{1}{24}$ articles cost..............$\frac{1}{15}$ of £1. 17*s.* 6*d.*
$1\frac{7}{12}=\frac{19}{12}=\frac{38}{24}$ articles cost $\frac{38}{15}$ of £1. 17*s.* 6*d.*

or,

$\frac{5}{8}$ articles cost ....£1. 17*s.* 6*d.*
$\frac{1}{8}$ articles cost..................$\frac{1}{5}$ of £1. 17*s.* 6*d.*
1 article cost ....$\frac{8}{5}$ of £1. 17*s.* 6*d.*
$\frac{1}{12}$ articles cost.........$\frac{1}{12}$ of $\frac{8}{5}$ of £1. 17*s.* 6*d.*
$1\frac{7}{12}=\frac{19}{12}$ articles cost $\frac{19}{12}$ of $\frac{8}{5}$ of £1. 17*s.* 6*d.*

Which may be condensed thus (see Ch. III. § 6):

$\frac{5}{8}$ articles cost ....£1. 17*s.* 6*d.*
1 article cost ....$\frac{8}{5}$ of £1. 17*s.* 6*d.*
$1\frac{7}{12}$ articles cost ......$\frac{19}{12}$ of $\frac{8}{5}$ of £1. 17*s.* 6*d.*

$$\frac{19}{\overset{}{\cancel{12}}_{3}} \times \frac{\overset{2}{\cancel{8}}}{\cancel{5}} \times \text{£}\cancel{1}.\ \overset{\overset{2s.\ 6d.}{\cancel{12s.\ 6d.}}}{\cancel{17s.\ 6d.}} = 19 \times 2 \times 2s.\ 6d. = \text{£}4.\ 15s.$$

Ans. £4. 15*s.*

Exercise XXV. (*d*).

(1) If $\frac{4}{5}$ of an article cost 3*s.* 9*d.*, what would $2\frac{3}{8}$ articles cost?

(2) If $2\frac{4}{7}$ guineas gain 4*s.* 6*d.*, what will be the profit on $3\frac{5}{6}$ guineas?

(3) If on a velocipede I travel $3\frac{3}{5}$ miles in $7\frac{1}{2}$ minutes, how many miles shall I travel in 50 minutes, and how long shall I take to travel 50 miles?

(4) If $25\frac{11}{50}$ francs are worth £1, how many francs shall I get for 16*s.*, and how much sterling money for a Napoleon (20 francs)?

(5) If a French ton is $\frac{49}{50}$ of an English ton, how many English tons, &c., are there in 700 French tons, and how many French tons in 700 English tons?

(6) If a stick $7\frac{1}{2}$ feet long throw a shadow of $8\frac{3}{4}$ feet, how high will the steeple be whose shadow is 50 yds. 6 in. long?

(7) A certain wheel makes 1760 turns in travelling a distance of 2 miles; how many turns would it make in travelling 1000 yards?

§ 5. If 17 men require $4\frac{2}{5}$ days to build a certain wall, how long will 13 men take?

Statement: 17 men require $4\frac{2}{5}$ days.
Question: 13 ,, x.

Mod. op.: 17 men require ....$4\frac{2}{5}$ days.
1 man requires ....$17 \times 4\frac{2}{5}$,,
13 men require ...$\frac{1}{13}$ of $17 \times 4\frac{2}{5}$,,

$\frac{1}{13} \times 17 \times 4\frac{2}{5} = \frac{374}{65} = 5\frac{49}{65}$. *Ans.* $5\frac{49}{65}$ days.

EXERCISE XXV. (*e*).

(1) If from a sack of flour I can make 45 loaves weighing 3 lbs. each, how many 4 lb. loaves can I make from it?

(2) If I have enough money to buy 50 articles at 3*s.* 6*d.* each, how many articles at 2*s.* 1*d.* each can I buy?

(3) If 30 bushels keep 50 horses for a week, how many horses would they keep for 25 days?

(4) If 30 bushels keep 50 horses for a week, for how long would they keep 15 horses?

(5) If 30 bushels keep 50 horses for a week, how many bushels would be required to keep 69 horses for the same time?

(6) How many yards at 5*s.* 6*d.* each must be given for 30 yards at 3*s.* 9*d.* each?

(7) The savings of a professional man are sufficient to purchase an annuity of £375, paying £80 cash for every £6. 10*s.* a year; how large an annuity will his money purchase if he waits till he has only to pay £65 cash for £6. 10*s.* a year?

§ 6. If 2 tons, 5 cwt. can be carried over 150 miles for 14*s.* 6*d.*, how far should 8 tons, 15 cwt. be carried for £1. 4*s.* 2*d.*?

Statement: 2 tons, 5 cwt. can be carried for 14*s.* 6*d.* over 150 miles.
Question: 8 tons, 15 cwt. „ £1. 4*s.* 2*d.* „ x „

Mod. op.:

2 tons, 5 cwt. (45 cwt.) can be carried for 14*s.* 6*d.* (174*d.*)......over 150 miles.
1 cwt. „ „ ......45×150 „
8 tons, 15 cwt. (175 cwt.) „ „ $\frac{1}{175}$ of 45×150 „
„ „ 1*d.* $\frac{1}{174}$ of $\frac{1}{175}$ of 45×150 „
„ £1. 4*s.* 2*d.* (290*d.*) $\frac{290}{174}$ of $\frac{1}{175}$ of 45×150 „

$$\frac{\overset{1}{\cancel{290}}}{\underset{1}{\cancel{174}}} \times \frac{\overset{3}{\cancel{45}}}{\underset{7}{\cancel{175}}} \times 150 = \frac{450}{7} = 64\tfrac{2}{7}.$$

Ans. $64\frac{2}{7}$ miles.

or thus:

2 tons, 5 cwt. (45 cwt.) can be carried for 14*s.* 6*d.* (174*d.*)......over 150 miles.
1 cwt. „ „ ......45×150 „
1 cwt. „ 1*d.* ....$\frac{1}{174}$ of 45×150 „
8 tons, 15 cwt. (175 cwt.) „ „ ...$\frac{1}{175}$ of $\frac{1}{174}$ of 45×150 „
„ £1. 4*s.* 2*d.* (290*d.*) $290 \times \frac{1}{175}$ of $\frac{1}{174}$ of 45×150 „
$290 \times \frac{1}{175} \times \frac{1}{174} \times 45 \times 150 = 64\frac{2}{7}$, as before.

EXERCISE XXV. (*f*).

(1) If 50 yards of calico 21 inches wide cost 11*s.* $5\frac{1}{2}d.$, what would $87\frac{1}{2}$ yards 27 inches wide cost?

(2) If 2 tons, 5 cwt. can be carried over 150 miles for 14*s.* 6*d.*, what weight can be carried a distance of 200 miles for £1. 4*s.* 2*d.*?

(3) If 12 spoons, each weighing 14 dwts., 14 grs., are worth £3. 5*s.* $7\frac{1}{2}d.$, what is the value of 21 spoons each weighing 1 oz.?

(4) If £275 gain £37. 10*s.* in 9 months, what should £990 gain in $2\frac{1}{2}$ months?

(5) Find the interest on £450 for 2 years at 4 per cent.

[Statement: Interest on £100 for 1 year is £4.]

(6) Find the interest on £760 for $1\frac{1}{2}$ years at $4\frac{1}{2}$ per cent.

(7) Find the interest on £45 for 8 months at $3\frac{3}{4}$ per cent.

(8) What sum of money will in $2\frac{1}{2}$ years at 5 per cent. yield £87. 10*s.* interest?

(9) At what rate per cent. will £75 yield £1. 15*s.* interest in 7 months?

(10) At what rate per cent. will £120 yield £3. 12*s.* interest in 146 days?

Exercise XXV. (*g*).

(1) If 750 men require 22,500 rations of food, how many rations will 1200 men require?

(2) If the clothing of 750 men costs £2831. 5*s.*, what will the clothing of 3500 men cost?

(3) If 7 cwt., 1 qr. cost £26. 10*s.* 4*d.*, what will 43 cwt., 2 qrs. cost?

(4) A garrison of 536 men has provisions to last from March 1st to Dec. 6th; how long will the provisions last if the garrison is increased by 588 men?

(5) What is the cost of 172 pieces of lead, each weighing 3 cwt., 2 qrs., $17\frac{1}{2}$ lbs., at £8. 17*s.* 6*d.* for $19\frac{1}{2}$ cwt.?

(6) If a wheel makes $2\frac{3}{4}$ turns in 1 minute and 17 seconds, how often will it revolve in 7 hours?

(7) If travelling at the rate of $12\frac{1}{2}$ miles an hour I require $15\frac{1}{2}$ hours to complete the journey, in how many hours shall I complete it if I increase the rate of travelling by $7\frac{1}{2}$ miles an hour?

(8) If £59. 10*s.* is required to buy an annuity of £8. 10*s.* a year, how much would be required to buy an annuity of £50?

(9) If with my money I can buy an annuity of £50 at the rate of £85 cash for every yearly £3. 10*s.*, what annuity will my money purchase if I pay £75 cash for every £3. 5*s.* a year?

(10) Find the interest on £548. 10*s.* 6*d.* at $4\frac{1}{2}$ per cent.

(11) What sum of money will at $5\frac{1}{2}$ per cent. produce the same interest that 1000 guineas produce at $3\frac{3}{4}$ per cent.?

(12) If in 1375 ounces of air there be $13\frac{3}{4}$ ounces of vapour, how much vapour would be contained in 1000 ounces of air?

(13) In what time will £524 at 5 per cent. yield £4. 7*s.* 4*d.* interest?

(14) If a house rated at £85 pays £4. 7*s.* 10*d.* rates, what will be paid upon a house rated at £245?

(15) In "quick" marching, soldiers take 110 steps of 30 inches per minute; at the "double" they take 150 steps of 36 inches per minute; if 1000 soldiers marching quick in fours take $\frac{1}{2}$ an hour to pass a house, how long will 720,000 men eighteen abreast take at the double?

CHAPTER V.

VARIOUS.

§ 1. What fraction of 50 is 20?

$1 = \frac{1}{50}$ of 20; $\therefore\ 20 = \frac{20}{50}\big|\frac{2}{5}$ of 50. *Ans.* $\frac{2}{5}$ of 50.

This answer means that 50 must be multiplied by $\frac{2}{5}$ to give 20. Hence the above question, What fraction of 50 is 20? means: By what must 50 be multiplied to yield 20? which is the question asked in division.

Hence, what fraction of 50 is $20 = 20 \div 50$, and generally.

" b is $a = a \div b$.

Answer, in every case, $a \div b$, or $\frac{a}{b}$. (Part II. Ch. III. § 7.)

Exercise XXVI. (*a*).

(1) What fraction of 8 is 3?
(2) " 3 is 8?
(3) " 9 is 5?
(4) " 5 is 9?
(5) " 12 is 9?
(6) " 9 is 12?
(7) " $1\frac{1}{5}$ is $\frac{2}{5}$?
(8) What fraction of $\frac{2}{5}$ is $1\frac{1}{5}$?
(9) " $1\frac{7}{8}$ is $\frac{5}{16}$?
(10) " $\frac{5}{16}$ is $1\frac{7}{8}$?
(11) " $3\frac{1}{5}$ is $\frac{8}{15}$?
(12) " £1 is 1*s.*?
(13) " £1 is 6*d.*?
(14) " £1 is 3*d.*?

(15) What fraction of £1 is 1*d.*?
(16) „ £1 is $\frac{1}{2}$*d.*?
(17) „ £1 is $\frac{1}{4}$*d.*?
(18) „ £1 is 16*s.*?
(19) „ £1 is 11*s.*?
(20) „ £1 is 2*s.* 10*d.*?
(21) „ £1 is 1*s.* 2$\frac{3}{4}$*d.*?
(22) „ £1 is 4*s.* 11$\frac{1}{4}$*d.*?
(23) „ £1 is 13*s.* 4*d.*?
(24) „ £1 is 17*s.* 10$\frac{3}{4}$*d.*?

What fraction of £3. 4*s.* 6*d.* is £2. 11*s.* 3*d.*?

£3*s.* 4*s.* 6*d.* = 774 pence; £2. 11*s.* 3*d.* = 615 pence; and 615 pence is $\frac{615}{774}$ of 774 pence. $\frac{615}{774}=\frac{205}{258}$. *Ans.* $\frac{205}{258}$.

What fraction of 5 tons, 8 cwt., 21 lbs., is 3 cwt., 17 lbs., 8 oz.?

5 tons, 8 cwt., 21 lbs. = 193872 oz.; 3 cwt., 17 lbs., 8 oz. = 5656 oz. 5656 oz. = $\frac{5656}{193872}$ of 193872 oz. $\frac{5656}{193872}=\frac{808}{27696}=\frac{101}{3462}$.
Ans. $\frac{101}{3462}$.

Reduce 3 years, 73 days, to the fraction of 1 year, 219 days. This means, What fraction of 1 year, 219 days, is 3 years, 73 days?

3 years, 73 days = 1168 days; 1 year, 219 days = 584 days. $\frac{1168}{584}=\frac{146}{73}=2$. *Ans.* 2.

What fraction of £$\frac{2}{11}$ is £$\frac{3}{7}$?

$\frac{3}{7}\div\frac{2}{11}=\frac{3}{7}\times\frac{11}{2}=\frac{33}{14}=2\frac{5}{14}$. *Ans.* $2\frac{5}{14}$.

Reduce $\frac{5}{7}$ of 1*s.* to the fraction of $8\frac{2}{5}$ guineas.

$\frac{5}{7}$ of 1*s.* $\div(8\frac{2}{5}\times 21s.)=\frac{5}{7}\div\frac{42}{5}$ of $21s.=\frac{5}{7}\div\frac{882}{5}=\frac{5}{7}\times\frac{5}{882}=\frac{25}{6174}$.
Ans. $\frac{25}{6174}$.

Reduce 1 oz. troy to the fraction of 1 oz. av.

1 oz. troy = 480 grs.; 1 oz. av. = $\frac{1}{16}$ of 7000 grs. = $\frac{7000}{16}$ grs. $480\div\frac{7000}{16}=480\times\frac{16}{7000}=\frac{192}{175}=1\frac{17}{175}$. *Ans.* $1\frac{17}{175}$.

EXERCISE XXVI. (*b.*)

(1) What fraction of £58 is £29?
(2) „ £29 is £58?
(3) „ £3. 17*s.* 4$\frac{1}{2}$*d.* is £1. 11*s.* 1$\frac{1}{2}$*d.*?
(4) „ £1. 11*s.* 1$\frac{1}{2}$*d.* is £3. 17*s.* 4$\frac{1}{2}$*d.*?
(5) „ £8. 10*s.* 10*d.* is 15*s.* 4*d.*?

(6)	What fraction of 15*s.* 4*d.* is £8. 10*s.* 10*d.*?			
(7)	,,	£3. 2*s.* $7\frac{1}{4}$*d.* is £1?		
(8)	,,	£1 is £3. 2*s.* $7\frac{1}{4}$*d.*?		
(9)	,,	2 tons, 13 cwt., is 1 ton, 1 cwt., 1 qr.?		
(10)	,,	2 lbs., 10 oz. av., is 1 lb., $4\frac{1}{2}$ oz. av.?		
(11)	,,	5 lbs., 9 oz. troy, is 6 oz., 15 dwts.?		
(12)	,,	2 yrs., 73 days, is 146 days?		
(13)	,,	5 tons, 8 cwt., 21 lbs., is 3 cwt., 17 lbs., 12 oz.?		
(14)	Reduce	£2. 10*s.* 6*d.*	to the fraction of	£1. 10*s.* 9*d.*
(15)	,,	6*s.* 7*d.*	,,	£8.
(16)	,,	2 lbs., 10 oz. av.	,,	2 lbs., 10 oz. troy.
(17)	,,	5 minutes	,,	1 day.
(18)	,,	$1\frac{1}{2}$*d.*	,,	1 guinea.
(19)	,,	$1\frac{1}{2}$ pints	,,	$2\frac{1}{2}$ gallons.
(20)	,,	1 lb. av.	,,	1 lb. troy.

§ 2. INTERPRETATION OF FRACTIONS.

We have found that the symbol $\frac{3}{4}$ bears the following interpretations:

(α) A quarter of *one* thing taken *three* times.

(β) A quarter of *three* things taken *once.*

(γ) $3 \div 4$.

(δ) The fraction that 3 is of 4.

(ϵ) The number by which 4 must be multiplied to give 3.

These are but different modes of expression for the same notion, and any one may be selected according to convenience. If the numerator and denominator are both abstract integers, they are all immediately intelligible.

Examine the symbol $\frac{4\frac{1}{5}}{2\frac{1}{3}}$. The interpretations ($\gamma$) ($\delta$) and ($\epsilon$) are readily intelligible, but (α) and (β) require an extension of language.

(α) may be given in these words: "Of equal pieces such that 4 of them make the whole, take 3." This wording applies to $\frac{4\frac{1}{5}}{2\frac{1}{3}}$, substituting $2\frac{1}{3}$ and $4\frac{1}{5}$ for 4 and 3 respectively.

(β) may be given thus: "Of equal pieces such that 4 of them make 3 wholes, take 1." This wording also applies to $\frac{4\frac{1}{5}}{2\frac{1}{3}}$ by making the above substitution.

We proceed to examine whether these five interpretations will
when applied to the symbol $\frac{4\frac{1}{5}}{2\frac{1}{3}}$, yield the same result.

(α) If $2\frac{1}{3}$ pieces make a unit, each piece must be $\frac{3}{7}$ of 1; for of $\frac{3}{7} = \frac{7}{3}$ of $\frac{3}{7} = 1$. If now we take $4\frac{1}{5}$ of such pieces, we obtain $\times \frac{3}{7} = \frac{21}{5} \times \frac{3}{7} = \frac{9}{5} = 1\frac{4}{5}$. *Ans.*

(β) If $2\frac{1}{3}$ pieces make $4\frac{1}{5}$ wholes, each piece must be $4\frac{1}{5}$ times great as if $2\frac{1}{3}$ pieces made 1; that is, $4\frac{1}{5} \times \frac{3}{7} = 1\frac{4}{5}$ as above. *Ans.*

(γ) $4\frac{1}{5} \div 2\frac{1}{3} = 4\frac{1}{5} \times \frac{3}{7} = 1\frac{4}{5}$. *Ans.*

(δ) What fraction of $2\frac{1}{3}$ is $4\frac{1}{5}$? 1 is $\frac{3}{7}$ of $2\frac{1}{3}$, $\therefore$ $4\frac{1}{5}$ is ($4\frac{1}{5} \times$ of $2\frac{1}{3}$; $4\frac{1}{5} \times \frac{3}{7} = 1\frac{4}{5}$. *Ans.*

(ϵ) By what number must $2\frac{1}{3}$ be multiplied to give $4\frac{1}{5}$? ((III. § 7.) $4\frac{1}{5} \div 2\frac{1}{3} = 1\frac{4}{5}$. *Ans.*

We thus see that the interpretation chosen will not affect result, whether the terms of the fraction be integral or fractio The interpretation (γ) is of easiest application, and therefore m generally adopted.

§ 3. Simplification of Fractions.

By simplification of a fraction, is meant finding the simplest p sible expression whose value is equal to that of the given fracti Hence the answer to a simplification of fractions ought always to either an integer, a proper fraction at lowest terms, or a mi number.

Simplify $\frac{8\frac{4}{7}}{1\frac{19}{21}}$.

$$\frac{8\frac{4}{7}}{1\frac{19}{21}} = 8\frac{4}{7} \div 1\frac{19}{21} = \frac{60}{7} \div \frac{40}{21} = \frac{\overset{3}{\cancel{60}}}{\underset{1}{\cancel{7}}} \times \frac{\overset{3}{\cancel{21}}}{\underset{2}{\cancel{40}}} = \frac{9}{2} = 4\frac{1}{2}. \qquad \textit{Ans.}$$

Simplify $\frac{\frac{5}{7} + \frac{1}{4}}{2\frac{1}{2} - 1\frac{6}{7}}$.

$$\frac{\frac{5}{7} + \frac{1}{4}}{2\frac{1}{2} - 1\frac{6}{7}} = \frac{\frac{20}{28} + \frac{7}{28}}{2\frac{7}{14} - 1\frac{12}{14}} = \frac{\frac{27}{28}}{\frac{9}{14}} = \frac{27}{28} \div \frac{9}{14} = \frac{\overset{3}{\cancel{27}}}{\underset{2}{\cancel{28}}} \times \frac{\overset{1}{\cancel{14}}}{\underset{1}{\cancel{9}}} = \frac{3}{2} = 1\frac{1}{2}. \qquad \textit{Ans.}$$

Simplify $\frac{\frac{14}{27} \times \frac{18}{35}}{1\frac{23}{25} \text{ of } \frac{5}{36}}$.

$$\frac{\frac{14}{27} \times \frac{18}{35}}{1\frac{23}{25} \text{ of } \frac{5}{36}} = \frac{\overset{\overset{1}{\cancel{2}}}{\cancel{14}}}{\underset{\underset{1}{\cancel{3}}}{\cancel{27}}} \times \frac{\overset{\overset{1}{\cancel{3}}}{\cancel{18}}}{\underset{\underset{1}{\cancel{7}}}{\cancel{35}}} \times \frac{\overset{\overset{1}{\cancel{5}}}{\cancel{25}}}{\underset{\underset{\underset{1}{\cancel{2}}}{\cancel{8}}}{\cancel{48}}} \times \frac{\overset{\overset{1}{\cancel{4}}}{\cancel{36}}}{\underset{1}{\cancel{5}}} = 1. \qquad \textit{Ans.}$$

Simplify $\dfrac{1}{6+\frac{1}{7\frac{1}{8}}}$

$\frac{1}{7\frac{1}{8}} = \frac{8}{57}$; $6 + \frac{8}{57} = \frac{350}{57}$; $\frac{1}{\frac{350}{57}} = \frac{57}{350}$. *Ans.* $\frac{57}{350}$.

Simplify $\dfrac{2\frac{3}{8}}{3\frac{1}{6}}$ of $\dfrac{4\frac{4}{9}}{\frac{1}{15}} \div \dfrac{8\frac{8}{9}}{7}$ of $\dfrac{4}{5\frac{5}{8}}$.

$$\frac{2\frac{3}{8} \times 4\frac{4}{9} \times 7 \times 5\frac{5}{8}}{3\frac{1}{6} \times \frac{1}{15} \times 8\frac{8}{9} \times 4} = \frac{\overset{1}{\cancel{19}} \times \overset{3}{\cancel{6}} \times \overset{1}{\cancel{40}} \times \overset{1}{\cancel{15}} \times \overset{1}{\cancel{9}} \times 7 \times 45}{\underset{4}{\cancel{8}} \times \underset{1}{\cancel{19}} \times \underset{1}{\cancel{9}} \times \underset{\underset{1}{\cancel{2}}}{\cancel{80}} \times 4 \times \underset{1}{\cancel{8}}} = \frac{945}{16} = 59\tfrac{1}{16}. \quad \textit{Ans. } 59\tfrac{1}{16}.$$

Simplify $\dfrac{(\frac{4}{11} \div \frac{3}{5}) + \frac{1}{11} \text{ of } 2 + \frac{1}{6} \times (\frac{1}{5} - \frac{1}{11})}{\frac{1}{4\frac{1}{2}} + \frac{1}{5\frac{1}{2}} + \frac{1}{7\frac{1}{2}}}$.

$\frac{4}{11} \div \frac{3}{5} = \frac{20}{33}$; $\frac{1}{11}$ of $2 = \frac{2}{11}$; $\frac{1}{6}$ of $(\frac{1}{5} - \frac{1}{11}) = \frac{1}{6}$ of $\frac{6}{55} = \frac{1}{55}$.

$$\frac{20}{33} + \frac{2}{11} + \frac{1}{55} = \frac{100+30+3}{165} = \frac{133}{165}.$$

$$\frac{1}{4\frac{1}{2}} = 1 \div \frac{9}{2} = \frac{2}{9}; \quad \frac{1}{5\frac{1}{2}} = \frac{2}{11}; \quad \frac{1}{7\frac{1}{2}} = \frac{2}{15}.$$

$$\frac{2}{9} + \frac{2}{11} + \frac{2}{15} = \frac{110+90+66}{495} = \frac{266}{495}.$$

$$\frac{133}{165} \div \frac{266}{495} = \frac{\overset{1}{\cancel{133}} \times \overset{\overset{3}{\cancel{99}}}{\cancel{495}}}{\underset{\underset{1}{\cancel{33}}}{\cancel{165}} \times \underset{2}{\cancel{266}}} = \frac{3}{2} = 1\tfrac{1}{2}. \qquad \textit{Ans. } 1\tfrac{1}{2}.$$

Exercise XXVII.

(1) $\dfrac{1}{3\frac{1}{3}}$

(2) $\dfrac{1}{3\frac{2}{3}}$

(3) $\dfrac{5}{3\frac{2}{3}}$

(4) $\dfrac{5\frac{1}{2}}{3\frac{2}{3}}$

(5) $\dfrac{3\frac{2}{3}}{5\frac{1}{2}}$

(6) $\dfrac{1\frac{1}{12}}{3\frac{7}{16}}$

(7) $\dfrac{4\frac{8}{27}}{2\frac{14}{81}}$

(8) $\dfrac{5\frac{13}{28}}{2\frac{19}{49}}$

(9) $\dfrac{2\frac{5}{8}}{8\frac{2}{5}} \times \dfrac{8\frac{2}{5}}{2\frac{5}{8}}$

(10) $\dfrac{2\frac{5}{8} \times 8\frac{2}{5}}{8\frac{2}{5} \div 2\frac{5}{8}}$

(11) $\dfrac{2\frac{5}{8} + 8\frac{2}{5}}{8\frac{2}{5} - 2\frac{5}{8}}$

(12) $\dfrac{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}}{\frac{1}{2\frac{1}{2}} + \frac{1}{3\frac{1}{2}} + \frac{1}{4\frac{1}{2}}}$

(13) $\dfrac{\frac{7}{8} + \frac{4}{9} + \frac{1}{10}}{\frac{1}{6} + \frac{1}{45} + \frac{1}{72}}$

(14) $\dfrac{\frac{7}{8} - (\frac{4}{9} + \frac{1}{10})}{\frac{1}{6} + \frac{1}{45} - \frac{1}{72}}$

(15) $\dfrac{1\frac{4}{17} \times 6\frac{4}{5}}{3\frac{3}{5} - 1\frac{2}{25}} + \frac{2}{11}$ of $(2\frac{2}{5} - \frac{3}{4}) - \dfrac{1\frac{3}{5}}{12}$

(16) $3\frac{1}{3} \div \dfrac{1 - \frac{15}{16}}{\frac{1}{6} - \frac{1}{8}}$

(17) $\cfrac{1}{2+\cfrac{1}{3\frac{1}{2}-\frac{7}{8}}}$

(18) $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}$

(19) $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}$

(20) $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}$

(21) $\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{15+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{2}}}}}}$

(22) $3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{25+\cfrac{1}{1+\cfrac{1}{7+\cfrac{1}{4}}}}}}}$

(23) $\dfrac{\frac{1}{3}+\frac{4}{9}+\frac{11}{27}}{\frac{1}{4}+\frac{5}{6}+\frac{11}{12}}$ of $\dfrac{\frac{5}{8} \text{ of } 2\frac{1}{3} \div 1\frac{11}{24}}{\frac{16}{27} \text{ of } 4\frac{5}{8} \div 9\frac{1}{4}}$

§ 4. G.C.M. and L.C.M. of Fractions.

The meaning of the expressions Measure and Multiple requires to be rendered somewhat more precise. If of two quantities the first is contained in the second an *integral* number of times, the first is a measure of the second, and the second a multiple of the first.

Examine $\frac{8}{3} \div \frac{2}{9}$. $\frac{8}{3} \div \frac{2}{9} = \frac{8}{3} \times \frac{9}{2}$.

$$\frac{\overset{4}{\cancel{8}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{3}{\cancel{9}}}{\underset{1}{\cancel{2}}} = 12.$$

This quotient is integral only because the 2 is a measure of the 8 and the 3 of the 9, as otherwise the denominators would not disappear. Generally: If $\frac{a}{b}$ and $\frac{c}{d}$ are two fractions at their lowest terms, $\frac{a}{b} \div \frac{c}{d}$ is integral only if c is a measure of a, and d a multiple of b.

In order, therefore, that one quantity should measure a series of quantities (of course at their lowest terms), its numerator must be a measure of each numerator, and its denominator a multiple of each denominator of the series. Hence G.C.M. of a series of fractions = G.C.M. of numerators ÷ L.C.M. of denominators; and conversely, L.C.M. of a series = L.C.M. of numerators ÷ G.C.M. of denominators. We may

also remark that the resulting G. C. M. or L. C. M. will be at its lowest terms; for if the G. C. M. of the numerators, say, have any factor, and that factor is contained in any one of the original denominators, one at least of the given fractions would not be at lowest terms.

Find G. C. M. and L. C. M. of $\frac{9}{35}$, $\frac{12}{25}$, $\frac{27}{50}$.

G. C. M. of 9, 12, 27, is 3. G. C. M. of 35, 25, 50, is 5.

L. C. M. of " is 108. L. C. M. of " is 350.

$\therefore$ G. C. M. required is $\frac{3}{350}$, and L. C. M. required is $\frac{108}{5} = 21\frac{3}{5}$.

EXERCISE XXVIII.

(1) Find G. C. M. and L. C. M. of $5\frac{1}{2}$, $7\frac{1}{3}$, $8\frac{1}{4}$, $4\frac{8}{9}$, $9\frac{1}{6}$, $6\frac{5}{12}$.

(2) " " $33\frac{3}{7}$ and $50\frac{5}{8}$.

(3) " " $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{12}$.

(4) " " $50\frac{1}{2}$, $67\frac{1}{3}$, $44\frac{8}{9}$, $84\frac{1}{6}$, 707.

(5) " " $225\frac{3}{5}$ and $181\frac{2}{7}$.

(6) " " $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$.

(7) " " $1\frac{1}{14}$, $1\frac{19}{21}$, $4\frac{2}{7}$, $\frac{25}{42}$.

§ 5. SURFACE MEASURE.

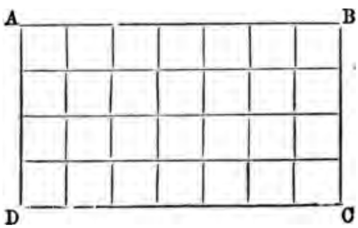
The extent of a surface expressed numerically is called its *area.*

A square surface which measures an inch each way is called a *square inch.*

A square surface which measures a foot each way is called a *square foot.* Similarly we have square yards, square miles, &c.

The area of a surface is expressed by the number of such square units that it contains,* or that would be required to cover it.

Find the area of a table 7 feet long and 4 feet broad.



* It will be seen that we only deal here with rectangular plane surfaces.

Let the line AB represent the length, which contains 7 *linear* feet, and AD the breadth, containing 4 linear feet. If lines be drawn as in the figure, the space ABCD will be divided into *square* feet, and there will be 4 rows of 7 square feet each, or 7 columns of 4 square feet each, i. e. $7 \times 4 = 28$ square feet.

Similar reasoning will in every case shew that the number of units in one side multiplied by the number of units in the other, will give the number of square units in the area.

Let l = the number of units in the length.
„ b = „ „ breadth.
„ a = „ square units „ area.

Then $l \times b = a$.

Exercise XXIX.

	From length,	breadth,	find area:
(1)	15 linear feet,	7 linear feet.	
(2)	10 „ yds.,	3 „ yds.	
(3)	5 „ in.,	2 „ in.	
(4)	1 „ foot,	1 „ in.	
(5)	2 „ yds.,	4 „ feet.	
(6)	12 „ in.,	12 „ in.	
(7)	3 „ feet,	3 „ feet.	
(8)	1760 „ yds.,	1760 „ yds.	

The following table will now be evident:

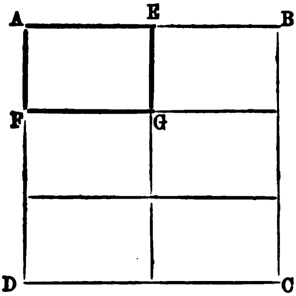
144 square inches = 1 square foot.
9 square feet = 1 square yard.

Since $a = l \times b$, if a and l are known, b can be found; for the question is, By what number must l be multiplied to give a? *Ans.* $a \div l$, or $\frac{a}{l}$. Hence, $\frac{a}{l} = b$; similarly, $\frac{a}{b} = l$.

Exercise XXX.

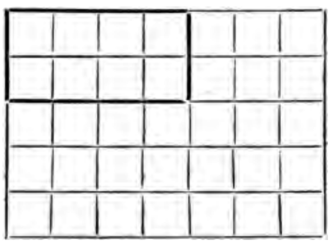
	From (or) Given } Area,	and one side,	find other side:
(1)	36 square yds.,	9 linear yds.	
(2)	28 „ in.,	7 „ in.	
(3)	100 „ feet,	5 „ feet.	
(4)	100 „ miles,	10 „ miles.	

Find the area if the length is $\frac{1}{2}$ ft. and the breadth $\frac{1}{3}$ ft.



Let AB = 1 foot, AE = $\frac{1}{2}$ ft., AD = 1 foot, AF = $\frac{1}{3}$ ft., then ABCD is 1 square foot, and AEGF is $\frac{1}{6}$ of 1 square foot, i.e. $\frac{1}{2} \times \frac{1}{3}$ square ft. Similar reasoning will shew in every case that if the length is $\frac{1}{a}$ and breadth $\frac{1}{b}$, the area is $\frac{1}{a} \times \frac{1}{b}$.

Next suppose length $\frac{4}{7}$ ft., breadth $\frac{2}{5}$ ft.



Examination of the diagram shews that, as before, length $\frac{1}{7}$ ft., breadth $\frac{1}{5}$ ft., gives $\frac{1}{35}$ sq. ft.; but length $\frac{4}{7}$, breadth $\frac{2}{5}$, gives $4 \times 2 \times \frac{1}{35} = \frac{8}{35}$, i.e. $\frac{4}{7} \times \frac{2}{5}$. Hence in every case, whether the given dimensions be fractional or integral, $l \times b = a$, and consequently $\frac{a}{b} = l$, and $\frac{a}{l} = b$.

Find the area if the length is 5 ft., 4 in., and breadth 2 ft., 9 in.

$$5 \text{ ft., } 4 \text{ in.} = 5\tfrac{1}{3} \text{ ft.}; \quad 2 \text{ ft., } 9 \text{ in.} = 2\tfrac{3}{4} \text{ ft.}$$

$$5\tfrac{1}{3} \times 2\tfrac{3}{4} = \frac{\overset{4}{\cancel{16}}}{3} \times \frac{11}{\underset{1}{\cancel{4}}} = \frac{44}{3} = 14\tfrac{2}{3} \text{ sq. ft.} = 14 \text{ sq. ft., } 96 \text{ sq. in.}$$

Or, 5 ft., 4 in. = 64 in.; 2 ft., 9 in. = 33 in.

$$64 \times 33 = 2112 \text{ sq. in.}$$

$$\begin{array}{r} 12\underline{)2112} \\ 12\underline{)176} \\ 14 \text{ sq. ft. and } 8\times 12 \text{ sq. in.} \end{array}$$

Ans. 14 sq. ft., 96 sq. in.

Required the length of carpet $\frac{3}{4}$ yd. wide, to cover a room 20 ft., 10 in. long, 15 ft., $10\frac{1}{2}$ in. broad.

The area of the floor is $\frac{20\frac{5}{6}}{3} \times \frac{15\frac{7}{8}}{3}$ sq. yds. The area of the carpet is the same; and as its width is known, the length is found by dividing the area by the width, $\frac{3}{4}$ yd.

$$\frac{20\frac{5}{6}}{3} \times \frac{15\frac{7}{8}}{3} \times \frac{4}{3} = \frac{125}{6\times 3} \times \frac{127}{\cancelto{2}{8}\times 3} \times \frac{\cancelto{1}{4}}{3} = \frac{15875}{324} = 48\frac{323}{324} \text{ yds.}$$

Exercise XXXI.

(1) Find the area, given:

a.	Length,	1 yd., 2 ft., 9 in.	breadth,	2 ft., 8 in.
b.	"	2 yds., 1 ft., $7\frac{1}{2}$ in.	"	$10\frac{1}{2}$ in.
c.	"	17 yds., 2 ft., 3 in.	"	1 yd., 1 ft., 10 in.
d.	"	10 yds., 2 ft., 11 in.	"	1 yd.
e.	"	1 ft., $1\frac{1}{2}$ in.	"	1 ft.
f.	"	36 yds.	"	4 yds., 1 ft., 5 in.
g.	"	$\frac{7}{11}$ yd.	"	$\frac{8}{9}$ yd.
h.	"	$\frac{7}{11}$ yd.	"	$\frac{8}{9}$ ft.
k.	"	$2\frac{5}{9}$ ft.	"	$5\frac{3}{8}$ in.
l.	"	$\frac{1}{2}$ in.	"	$\frac{1}{2}$ in.

(2) A room is 20 ft. long, 14 ft., 6 in. broad, and 11 ft., 9 in. high. How many square feet of carpet will be required, and how many of paper?

(3) What length of carpet, 2 ft., 3 in. wide, and of paper, 1 ft., 9 in. wide, will be required?

(4) How many acres in a square furlong?

(5) A rod or pole is $5\frac{1}{2}$ yds. How many square yards in a square rod?

(6) Express in acres the difference between half a square mile and half a mile square.

(7) Find the difference between 2 square furlongs and 2 furlongs square.

(8) The difference between $3\frac{1}{8}$ square yards and $3\frac{1}{8}$ yards square.

(9) Find the one side, given :

a.	Area,	50 square ft.	the other side,	$3\frac{1}{2}$ yds.
b.	„	$185\frac{2}{3}$ square ft.	„	4 yds., 2 ft., $7\frac{1}{2}$ in.
c.	„	$6\frac{1}{2}$ square ft.	„	$10\frac{2}{5}$ ft.

(10) How many pieces of paper, 12 yds. long and 21 in. wide, will paper a room $5\frac{1}{4}$ yds. wide, $7\frac{1}{2}$ yds. long, $8\frac{3}{4}$ ft. high, and what will be the cost at 2*s.* $10\frac{1}{2}$*d.* per piece?

(11) Find the difference in expense between carpeting a room 18 ft. long, 13 ft., 8 in. wide, with Brussels, 27 in. wide, at 5*s.* 6*d.* a yard, and with Kidderminster a yard wide at 3*s.* 9*d.* a yard.

(12) Find the cost of 9 venetian blinds, 7 ft., 10 in. long, and 4 ft. $7\frac{1}{2}$ in. wide, at $8\frac{1}{2}$*d.* per square foot.

§ 6. Solid Measure.

The space filled by a body, expressed numerically, is called its *volume*, or cubic content.

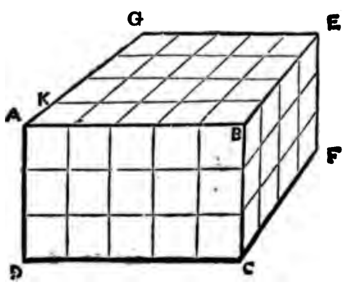
A cube which measures an inch each way is called a *cubic inch.*

A cube which measures a foot each way is called a *cubic foot.*

Similarly we have cubic yards, &c.

The volume of a body is expressed by the number of such cubic units that it contains, or that would fill the same amount of space.

Find the volume of a block of marble, 5 ft. long, 4 ft. broad, and 3 ft. thick.



Let AB be the length, 5 feet; AD the thickness, 3 feet; AG the breadth, 4 feet.

The area of the face, ABCD $= 5 \times 3 = 15$ square feet. If the breadth were 1 foot, as AK, the block would yield 15 cubic feet; but being 4 feet broad, the block yields $4 \times 15 = 60$ cubic feet.

Similar reasoning will in every case shew that the number of square units in one face must be multiplied by the number of linear units in the side independent of that face. Let l, b, t, be the number of units in the length, breadth and thickness respectively; $l \times b$, $b \times t$, $t \times l$, are the areas of the several faces; and $l \times b \times t = v$, v being the volume.

The following table will now be evident:

1728 cubic inches = 1 cubic foot.
27 cubic feet = 1 cubic yard.

Exercise XXXII.

	From length,	breadth,	thickness,	find volume:
(1)	8 ft.	5 ft.	3 ft.	
(2)	10 in.	10 in.	10 in.	
(3)	1 yd.	1 ft.	1 in.	
(4)	$2\frac{1}{2}$ yds.	$1\frac{3}{4}$ yds.	$\frac{5}{8}$ yds.	
(5)	10 yds., 1 ft.	2 ft., 8 in.	1 ft., $10\frac{1}{2}$ in.	
(6)	1 mile	15 yds.	$10\frac{1}{2}$ ft.	

Since $v = l \times b \times t$, $\frac{v}{l \times b} = t$; $\frac{v}{l \times t} = b$; $\frac{v}{t \times b} = l$; $\frac{v}{l} = b \times t$; $\frac{v}{b} = t \times l$; $\frac{v}{t} = l \times b$.

Exercise XXXIII.

(1) Given volume,	length,	and breadth, find thickness:
a. 120 cub. ft.	8 ft.	5 ft.
b. 5760 cub. in.	1 yd.	1 ft., 8 in.
c. 9 cub. yds., 7 ft., 1512 in.	6 yds., 7 in.	2 ft., 3 in.
d. 1 cub. yd.	4 yds.	2 yds., 1 ft.
e. $4\frac{7}{8}$ cub. ft.	$3\frac{1}{3}$ yds.	$2\frac{1}{4}$ in.

(2) Given volume,	area of one face, find the third side:
a. 60 cub. ft.	12 sq. ft.
b. 5 cub. yds., 25 ft.	17 sq. yds., 7 ft.
c. $8\frac{1}{15}$ cub. yds.	$1\frac{2}{9}$ sq. ft.
d. $2\frac{1}{4}$ cub. in.	150 sq. yds.

(3) Given volume and one side, find area of one face:

a. 343 cub. yds.	7 yds.
b. 1 cub. yd.	$1\frac{1}{2}$ in.
c. $3\frac{3}{8}$ cub. in.	$\frac{1}{1000}$ in.
d. $1\frac{2}{5}$ cub. ft.	1 yd.

(4) How many bricks are there in a wall, 7 yds., 2 ft., 4 in. long, 2 yds., 2 ft., 3 in. high, 1 ft., $1\frac{1}{2}$ in. thick? A brick is 9 in. long, $4\frac{1}{2}$ in. broad, 3 in. thick.

(5) A pile of stones is 12 yds. long, $4\frac{1}{2}$ yds. broad, 5 ft. high. How many stones are there, each being 1 foot long, $4\frac{1}{2}$ in. broad, 5 in. thick?

(6) How many cubic feet of wood are there in a block of timber, 15 ft., $10\frac{1}{2}$ in. long, 2 ft., $7\frac{1}{2}$ in. broad, and 1 ft., 2 in. thick?

(7) Gold can be beaten out to the thickness of $\frac{1}{200000}$ of an inch. How much surface can a cubic inch of gold be made to cover?

(8) If $\frac{2}{5}$ of a cubic foot of gold is beaten out to cover 1 acre, find its thickness.

(9) Find the value of a mass of timber, 6 yds., 2 ft., 4 in. long, 1 ft., 9 in. broad, 10 in. thick, at $7\frac{3}{4}d.$ per cubic foot.

(10) Find the area of the reservoir that would contain the daily water supply of London, 83361824 gallons, if each gallon has $277\frac{11}{40}$ cubic inches, supposing it to be 6 feet deep?

(11) Find the cost of lining with tin a cubical box, one edge of which is 4 ft., 6 in., at 1*s.* 6*d.* per square yard.

EXERCISE XXXIV.

Miscellaneous Examples on Vulgar Fractions.

(1) Find the value of:

a. $\frac{15}{28}$ of £2. 13*s.* $1\frac{1}{4}d.$

b. $\frac{15}{28}$ of 7 cwt., 3 qrs., 9 lbs.

c. $\frac{15}{28}$ of 2 years, 73 days.

(2) Find the value of $4\frac{4}{9}$ guineas + $3\frac{3}{5}$ of 2*s.* 6*d.* + £$\frac{7}{15}$ − $\frac{3}{5}$ of 1*s.*

(3) Simplify $(7\frac{5}{12} + 6\frac{5}{8}) + (7\frac{5}{12} - 6\frac{5}{8}) + (7\frac{5}{12} \text{ of } 6\frac{5}{8}) - (7\frac{5}{12} \times 6\frac{5}{8}) + [(7\frac{5}{12} \div 6\frac{5}{8}) \times (6\frac{5}{8} \div 7\frac{5}{12})]$.

(4) Interpret the symbol $\times$, and apply it, where possible, to the following:

a. £7×3.	*c.* £7×0.	*e.* £$\frac{3}{7} \times \frac{5}{9}$.	*g.* $\frac{3}{7} \times$ £$\frac{5}{9}$.
b. £7×1.	*d.* £$\frac{2}{5} \times \frac{1}{2}$.	*f.* £$\frac{5}{9} \times \frac{3}{7}$.	*h.* £$\frac{3}{7} \times$ £$\frac{5}{9}$.

(5) What fractions of £1 are 13*s.* 8*d.*, 17*s.* $10\frac{1}{2}$*d.*, 2*s.* $3\frac{1}{4}$*d.*?

(6) Divide $5\frac{3}{8}$ of 2*s.* 6*d.* by $\frac{3}{7}$ of 1*s.*

(7) A can do a piece of work in 4 days, and B in 5 days. In what time will they do it together?

[A in 1 day does $\frac{1}{4}$ of the work;
B " $\frac{1}{5}$ of the work;
∴ A and B together " do $\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$ of the work; as many times, then, as $\frac{9}{20}$ of the work is contained in the whole work, so many days will A and B together require. $1 \div \frac{9}{20} = \frac{20}{9} = 2\frac{2}{9}$.]

(8) A can do a piece of work in 12 days, which B can do in 10 and C in 15 days. In what time will all three together do it?

(9) A can do a piece of work in 14 days; B works twice as fast as A, and C can do it in 10 days. In what time can all three together do it?

(10) A can mow 2 acres in 7 days, B can mow 3 acres in 10 days. In what time will the two together mow 1 acre?

(11) A and B together can do a piece of work in 8 days, A alone can do it in 12 days. In what time could B alone do it?

(12) Simplify $\dfrac{\frac{2\frac{1}{2}}{1} + 1\frac{2}{3} + \frac{1}{3\frac{1}{2}}}{3\frac{2}{3} - 2\frac{1}{4}}$.

(13) Express $\frac{1}{29}$ of 1 lb. troy + $\frac{1}{29}$ of 1 lb. av. both as troy and as av. weights.

(14) Multiply $49\frac{15}{16}$ by $50\frac{1}{16}$, and add $\frac{1}{256}$ to the result.

(15) Reduce $\frac{3503}{4294}$ to lowest terms.

(16) Arrange in descending order of magnitude, $\frac{13}{25}$, $\frac{15}{26}$, $\frac{16}{27}$.

(17) If the step of a man be $2\frac{1}{3}$ feet, and that of a horse $2\frac{3}{4}$ feet, how many horse-paces are equal to 50 man-paces?

(18) Find the smallest exact number of feet which is an exact number of horse-paces and of man-paces.

(19) Find the largest quantity which is contained a whole number of times in each of the following: $2\frac{5}{9}$, $6\frac{7}{18}$. $11\frac{1}{2}$, $19\frac{1}{6}$.

(20) Find the value of £$\frac{1}{1000}$ + $\frac{1}{100}d$.

(21) The area of a certain room is $265\frac{5}{6}$ square feet; its length is $17\frac{3}{8}$ feet. Find its breadth.

(22) The volume of a log of wood is 115 cubic feet; its breadth is $3\frac{3}{8}$ feet; its thickness, $11\frac{1}{2}$ inches. Find the length.

(23) What fraction of 12*s.* 6*d.* must be added to $\frac{5}{7}$ of 1 guinea to make £1?

(24) Multiply the sum of $\frac{2}{3}$ of £100, and $\frac{1}{2\frac{1}{6}}$ of £6. 6*s.* 8*d.*, by $\frac{2}{7}$ of $\frac{\frac{5}{6}+\frac{3}{5}}{1\frac{1}{28}}$.

(25) Find L.C.M. of $\frac{3}{14}$, $1\frac{9}{35}$, $\frac{33}{56}$, $3\frac{20}{63}$.

(26) On Monday I spent $\frac{3}{8}$ of my money, on Tuesday $\frac{2}{5}$ of the original sum, and had then £11. 12*s.* 6*d.* left. How much had I at first?

(27) If on Tuesday I had spent $\frac{2}{5}$ of what was left me from Monday, and had then had £11. 12*s.* 6*d.* left, what would my original sum have been?

(28) Simplify $\frac{2\frac{2}{3}+1\frac{1}{2}}{2\frac{2}{3}-1\frac{1}{2}} \div \frac{5}{8}$.

(29) A person bequeathed $\frac{5}{12}$ of his property to A, $\frac{1}{4}$ of it to B, $\frac{1}{6}$ to C, and $\frac{1}{8}$ to D. The remainder, £550, to charities. What was the value of the whole property?

(30) If 1 bushel last $3\frac{2}{7}$ days, how many days will $4\frac{11}{20}$ bushels last?

(31) Divide the sum of $8\frac{4}{7}$ and $4\frac{4}{9}$ by

a. the sum
b. the difference } of their reciprocals.
c. the product

(32) Machine A can pump 3 gallons in 5 minutes, machine B works half as fast again as A, and C at half B's speed. In what time would A alone pump 1 gallon?

B	"	4 gallons?
C	"	$1\frac{1}{2}$ gallons?
A, B and C together		$2\frac{5}{8}$ gallons?

(33) Divide the sum of $\frac{2}{3}$ of $3\frac{3}{10}$, $\dfrac{1\frac{3}{4}}{2\frac{5}{8}}$ of 17, and $\frac{3}{5}$ of $5\frac{3}{4}$ of $\frac{34}{51}$, by 19.

(34) Simplify $\frac{15}{17}$ of $\dfrac{6\frac{4}{5}}{2\frac{8}{11}}$ of $(2\frac{7}{9} \div 3\frac{8}{9})$.

(35) Express $\frac{1}{5}$ of 13*s.* 4*d.* as a fraction of £5.

(36) If $\frac{3}{7}$ of an estate be worth £450, what is the worth of $\frac{11}{24}$ of it?

$$\left[\begin{array}{lcl} \frac{3}{7} & \text{is worth} & £450; \\ 1 & ,, & \frac{7}{3} \text{ of } £450; \\ \frac{11}{24} & ,, & \frac{11}{24} \times \frac{7}{3} \times £450. \end{array}\right]$$

(37) A can do a piece of work in 10 days, B can do it in 12 days, and C in 9 days. In what time will all three do $2\frac{1}{2}$ such pieces of work? What share of all the work is done by A, B and C respectively? If £1. 10*s.* 11*d.* is paid per piece of work, how much should each receive for the $2\frac{1}{2}$ pieces?

(38) A man can do a piece of work in 5 days, which a woman would take 8 and a boy 12 days to finish; the man worked $1\frac{1}{4}$ days and was joined by the woman; both together then worked for $1\frac{1}{2}$ days, leaving the remainder to be finished by the boy. How long will he take to complete his task, and which of the three will have done the largest and which the smallest share of the work?

(39) Give three different interpretations to the symbol $\frac{3}{4}$, and apply each to $\dfrac{4\frac{1}{5}}{6\frac{1}{3}}$.

(40) I had $\frac{2}{3}$ of a ship and sold $\frac{4}{5}$ of this share for £1200. What is the value of the whole ship?

(41) If 220 gals. of creosote, at 1*d.* per gal., give as much heat as $2\frac{1}{2}$ tons of coal, what will be the cost of the quantity of creosote that has the heating power of 1 ton of coal?

(42) The value of an oz. of standard gold is £3. 17*s.* $10\frac{1}{2}$*d.* What fraction of £1000 are 625 oz. of gold?

(43) Find the value of $\frac{2}{3}$ of $\frac{5}{8}$ of $\dfrac{1}{2\frac{1}{2}}$ of 7 articles, if $\frac{3}{5}$ of $1\frac{3}{4} \times 2\frac{1}{3}$ articles cost £15. 7*s.* 8*d.*?

(44) A cistern of 960 gallons is emptied by two pipes, A and B, in 5 and 7 minutes respectively. How much water will pass through each pipe if both are opened together?

(45) A and B together can do a piece of work in 6 days, A and

C in 8 days, B and C in 9 days. In what time could all three together do it?

(46) A brick is 9 in. long, $4\frac{1}{2}$ in. wide, and 3 in. thick. How many bricks are wanted to build a wall 520 yds., 9 in. long, 15 ft. high and $1\frac{1}{2}$ ft. thick?

(47) What will be the cost of painting the four walls of a room which is 24 ft., 3 in. long, 11 ft., 9 in. broad, and 11 ft., 6 in. high, at 1*s.* 6*d.* per sq. ft.?

(48) Find the average of $21\frac{2}{3}$, $73\frac{4}{5}$, 0, $3\frac{13}{200}$, 82, $17\frac{3}{20}$, $5\frac{1}{4}$, $9\frac{5}{12}$.

(49) If I spend on the first day $\frac{3}{5}$ of my money, next day $\frac{3}{5}$ of what is left, and so on for 4 days, what fraction of the original sum will be left?

(50) By what must the difference between $\frac{51}{85}$ and $\frac{13}{55}$ be

$$\left.\begin{array}{l} a.\ \text{increased,} \\ b.\ \text{multiplied,} \\ c.\ \text{divided,} \end{array}\right\} \text{ to give 12?}$$

(51) Divide 15*s.* 9*d.* by $2\frac{5}{8}$.

(52) Divide 15*s.* 9*d.* by 2*s.* $7\frac{1}{2}$*d.*

(53) Divide 15*s.* 9*d.* by £$2\frac{5}{8}$.

(54) A cistern can be filled by pipes A and B in 5 and 6 minutes respectively, and emptied by C in 4 minutes. In what time will the cistern be filled if all three are opened?

(55) From 1 lb. troy are coined $46\frac{29}{40}$ sovs. Express the weight of 1 sov., both by troy and av. weight.

(56) Simplify $$\frac{1\frac{13}{20}\times 4+\dfrac{2\frac{1}{40}}{1\frac{4}{5}}+\left(\frac{3}{20}\div\frac{3}{22}\right)}{\dfrac{(4\frac{1}{5}\times 6\frac{2}{3})+(11\frac{1}{5}\times 1\frac{2}{3})}{11\frac{1}{5}\times 6\frac{2}{3}}}\times\frac{2\frac{1}{3}}{1\frac{5}{8}} \text{ of } \frac{1\frac{5}{8}}{2\frac{1}{3}}$$

(57) Simplify $$\frac{1}{6+\frac{1}{7\frac{1}{3}}}+\frac{3}{5} \text{ of } 1\frac{2}{5}+\left(1\frac{2}{7}\div 1\frac{1}{4}\right)+\frac{9}{350}.$$

(58) The area of a certain floor is $145\frac{1}{2}$ sq. ft.; its length is 15 ft., $4\frac{1}{2}$ in. Find the width.

(59) A court-yard is to be paved with tiles $10\frac{1}{2}$ in. square. How many tiles will be wanted if the court is 7 yds., 2 ft., 4 in. long, and *4 yds., 2 feet* wide?

(60) How many tiles would have been wanted if each had an area of $10\frac{1}{2}$ sq. in.?

(61) State and prove the rules for the multiplication and division of one vulgar fraction by another. Shew that the multiplication of two proper fractions will give a product less than either of them.

(62) Simplify

α. $\dfrac{m \times a}{b \times m}$. ζ. $\dfrac{m \times a + m \times b}{m}$.

β. $\dfrac{a}{b \times c} + \dfrac{b}{c \times a} + \dfrac{c}{b \times a}$. η. $\dfrac{1}{a + \frac{1}{b}}$.

γ. $\dfrac{a}{b} \times \dfrac{c}{d}$. θ. $\left(\dfrac{a}{b} + \dfrac{c}{d}\right) \times b$.

δ. $\dfrac{a}{b} \div \dfrac{c}{d}$. ι. $\left(\dfrac{a}{b} - \dfrac{c}{d}\right) \times d$.

ϵ. $\dfrac{a}{b} \times \dfrac{c}{d} \times b \times d$. κ. $\left(\dfrac{a}{b} \pm \dfrac{c}{d}\right) \times b \times d$.

(63) Simplify $\dfrac{\frac{27}{64} - \frac{8}{27}}{\frac{9}{16} + \frac{1}{2} + \frac{4}{9}} + \dfrac{4\frac{1}{7} - 2\frac{1}{4}}{6\frac{1}{2} - 2\frac{1}{7}}$.

(64) Divide 1 lb. troy and also 1 lb. avoirdupois by 17 lb. 10 oz., 6 dwts., 15 grs. troy.

(65) A room is 8 yds., 2 ft., 3 in. long, and 5 yds., 9 in. broad. Find the cost of covering it with carpet $\frac{3}{4}$ yd. wide, at 4*s.* 6*d.* a yd.

(66) If the value of one rupee is $1\frac{23}{24}$*s.* how many rupees can be bought for £$7\frac{14}{15}$?

(67) If 56 cubic feet, 1044 cubic inches, of timber are required to floor a room $29\frac{1}{4}$ feet long, and $25\frac{1}{3}$ feet wide, what is the thickness of the boards?

(68) Find the fraction of £1 which is equivalent to the excess of $\frac{2}{3}$ of a guinea above the sum of $\frac{3}{4}$ of 1*s.* and $\frac{4}{9}$ of 7*s.* 6*d.*

(69) The cargo of a ship worth £45000 belongs to three partners; A owns $\frac{7}{9}$ of $\frac{3}{5}$ of it; B's share is equal to $3\frac{3}{14}$ times $\frac{2}{9}$ of A's share, and C owns the remainder. What part of the cargo is owned by each partner, and what ought each to receive from the sale?

(70) If the circumference of a wheel is $3\frac{1}{7}$ times its diameter, how many times will a wheel 1 yd., $1\frac{1}{8}$ ft. in diameter, revolve in travelling $3\frac{3}{7}$ miles?

CHAPTER VI.

PRACTICE.

§ 1. Practice is another mode of performing multiplication.

CASE I. Find the cost of 428 articles at 2*s.* each.

$2s. = \frac{1}{10}$ of £1, ∴ $428 \times 2s. = £\frac{428}{10} = £42\frac{8}{10} =$ £42. 8 florins = £42. 16*s.* Similarly $3259 \times 2s.$ = £325. 9 florins = £325. 18*s.*, and any number of times 2*s.* may be obtained by doubling the last figure for the shillings, and calling the rest pounds.

EXERCISE XXXV.

(1) $371 \times 2s.$
(2) $8643 \times 2s.$
(3) $615 \times 2s.$
(4) $8497 \times 2s.$
(5) $1729 \times 2s.$
(6) $4572 \times 2s.$
(7) $9724 \times 2s.$
(8) $1376 \times 2s.$
(9) $7948 \times 2s.$
(10) $320 \times 2s.$

CASE II. Find the cost of 367 articles at 12*s.* each.

12*s.* = 6 florins, $367 \times 12s. = (367 \times 1$ fl.$) \times 6 =$ (£36. 7 fl.$) \times 6 =$ £216. 42 fl. = £220. 2 fl. = £220. 4*s.*

Wording: **36,7 × 6 = 42 (florins) 4′, carry 4; 36, 40′, carry 4; 18, 22′.**

$5493 \times 14s.$ $549,3 \times 7 =$ £3845. 2*s.*

Wording: **21, 2′, carry 2; 63, 65′, carry 6; 28, 34′, carry 3; 35, 38′.**

EXERCISE XXXVI.

(1) $4378 \times 12s.$
(2) $987 \times 14s.$
(3) $6716 \times 16s.$
(4) $3545 \times 18s.$
(5) $3714 \times 8s.$
(6) $7341 \times 4s.$
(7) $8267 \times 6s.$
(8) $9752 \times 8s.$
(9) $723 \times 12s.$
(10) $8769 \times 10s.$

CASE III. Find the cost of 1389 articles at 1*s.* each.

$1389 \times 1s. = \frac{1}{2}$ of $1389 \times 2s. = \frac{1}{2}$ of £138. 9 fl. = £69. 9*s.*

Wording: **2 in 13, 6′, carry 1; in 18, 9′; in 9 fl., 9′s.**

$7356 \times 1s. = \frac{1}{2}$ of £735. 6 fl. = £367. 16*s.*

Wording: **2 in 7, 3′, carry 1; in 13, 6′, carry 1; in 15, 7′, carry 1; in 16 fl., 16′s.**

EXERCISE XXXVII.

(1) $9802 \times 1s.$
(2) $8491 \times 1s.$
(3) $4624 \times 1s.$
(4) $3765 \times 1s.$
(5) $537 \times 1s.$
(6) $1280 \times 1s.$
(7) $1579 \times 1s.$
(8) $658 \times 1s.$
(9) $713 \times 1s.$
(10) $5046 \times 1s.$

CASE IV. Find the cost of 4769 articles at 17*s.* each.

$4769 \times 16s. =$	£3815	4	0
,, $\times 1s. =$	238	9	0
,, $\times 17s. =$	£4053	13	0

EXERCISE XXXVIII.

(1) $4253 \times 3s.$
(2) $8674 \times 7s.$
(3) $2587 \times 9s.$
(4) $483 \times 13s.$
(5) $5724 \times 17s.$
(6) $3048 \times 19s.$
(7) $956 \times 11s.$
(8) $5010 \times 17s.$
(9) $3466 \times 19s.$
(10) $8888 \times 3s.$

CASE V. ALIQUOT PARTS. An aliquot part of any quantity is a measure of that quantity. Thus 6*s.* 8*d.*, being $\frac{1}{3}$ of £1, is an aliquot part of £1. Similarly $1\frac{1}{2}d.$ ($\frac{1}{4}$ of 6*d.*) is an aliquot part of 6*d.* 21 lbs. ($\frac{1}{16}$ of 3 cwts.) is an aliquot part of 3 cwts. The useful aliquot parts of £1 are 10*s.*, 6*s.* 8*d.*, 5*s.*, 4*s.*, 3*s.* 4*d.*, 2*s.* 6*d.*, 2*s.*, 1*s.* 8*d.*, 1*s.*

Find the cost of 419 articles at 3*s.* 4*d.* each (3*s.* 4*d.* $= \frac{1}{6}$ of £1.)
$419 \times 3s.\ 4d. = £\frac{419}{6} = £69.\ 16s.\ 8d.$

EXERCISE XXXIX.

(1) $846 \times 10s.$
(2) $789 \times 4s.$
(3) $8736 \times 2s.\ 6d.$
(4) $1234 \times 1s.\ 8d.$
(5) $477 \times 3s.\ 4d.$
(6) $3124 \times 5s.$
(7) $385 \times 6s.\ 8d.$
(8) $4457 \times 2s.\ 6d.$
(9) $1494 \times 3s.\ 4d.$
(10) $779 \times 6s.\ 8d.$
(11) $387 \times 10s.$
(12) $7773 \times 2s.\ 6d.$
(13) $765 \times 1s.\ 8d.$
(14) $1452 \times 2s.\ 6d.$
(15) $17 \times 10s.$
(16) $2831 \times 5s.$
(17) $43 \times 6s.\ 8d.$
(18) $787 \times 3s.\ 4d.$
(19) $29 \times 5s.$
(20) $76047 \times 3s.\ 4d.$
(21) $451 \times 2s.\ 6d.$
(22) $4284 \times 6s.\ 8d.$
(23) $2318 \times 5s.$
(24) $51 \times 4s.$
(25) $1717 \times 5s.$
(26) $11 \times 3s.\ 4d.$

(27) $1837 \times 4s.$
(28) $2655 \times 3s.\ 4d.$
(29) $2470 \times 2s.\ 6d.$
(30) $4776 \times 1s.\ 8d.$
(31) $38 \times 2s.\ 6d.$
(32) $588 \times 1s.\ 8d.$
(33) $848 \times 3s.\ 4d.$
(34) $3786 \times 2s.\ 6d.$
(35) $567 \times 2s.\ 6d.$
(36) $11111 \times 1s.\ 8d.$

CASE VI. The useful aliquot parts of 2*s.* are 1*s.*, 8*d.*, 6*d.*, 4*d.*, 3*d.*, 2*d.*; those of 1*s.* are 6*d.*, 4*d.*, 3*d.*, 2*d.*, $1\frac{1}{2}d.$ and 1*d.*; of 6*d.* are 3*d.*, 2*d.*, $1\frac{1}{2}d.$, 1*d.*, $\frac{3}{4}d.$, $\frac{1}{2}d.$

Find the cost of 477 articles at 8*d.* each.

$477 \times 8d. = \frac{1}{3}$ of £47. 7 fl. = £15. 18*s.*

Wording: 3 in 4, **1′**, carry 1; in 17, **5′**, carry 2; in 27 fl., 9 fl. = **18′s.**

Find the cost of 643 articles at 8*d.* each.

$64_{\iota}3 \times 8d.$ = £21. 8*s.* 8*d.*

Wording: 3 in 6, **2′**; in 4, **1′**, carry 1; (13 fl. = 26*s.*); in 26, **8′**, carry 2*s.*; in 24, **8′**.

Find the cost of 5261 articles at 8*d.* each.

$526_{\iota}1 \times 8d.$ = £175. 7*s.* 4*d.*

N.B. Compare the wording of the first with that of the two following examples here given. In the first the number of florins is divisible by 3.

EXERCISE XL.

(1) $5430 \times 8d.$
(2) $5124 \times 2d.$
(3) $1524 \times 6d.$
(4) $1740 \times 6d.$
(5) $3432 \times 4d.$
(6) $41296 \times 3d.$
(7) $3456 \times 8d.$
(8) $26413 \times 2d.$
(9) $5899 \times 6d.$
(10) $4106 \times 3d.$
(11) $2745 \times 4d.$
(12) $12113 \times 3d.$
(13) $5468 \times 2d.$
(14) $6794 \times 8d.$
(15) $8142 \times 6d.$
(16) $8182 \times 2d.$
(17) $4533 \times 6d.$
(18) $2330 \times 4d.$
(19) $6658 \times 8d.$
(20) $4507 \times 4d.$
(21) $8155 \times 3d.$
(22) $7346 \times 6d.$
(23) $3751 \times 2d.$
(24) $4414 \times 4d.$
(25) $9255 \times 3d.$
(26) $7646 \times 8d.$
(27) $5273 \times 2d.$
(28) $2851 \times 8d.$
(29) $7108 \times 3d.$
(30) $6498 \times 2d.$
(31) $2537 \times 4d.$
(32) $9059 \times 6d.$
(33) $1439 \times 2d.$
(34) $3413 \times 6d.$
(35) $17222 \times 3d.$
(36) $21329 \times 3d.$

CASE VII. Find the cost of 1749 articles at £3. 7*s.* 9¾*d.* each.

A	1749 × £3. =	£5247	0	0
B	„ × 6*s.* =	524	14	0
C	„ × 1*s.* =	87	9	0
D	„ × 6*d.* =	43	14	6
E	„ × 3*d.* =	21	17	3
F	„ × ¾*d.* =	5	9	3¾
		£5930	4	0¾

D is obtained by halving C,
E „ „ D,
F „ finding the quarter of E.

Find the cost of 2359 articles at £1. 14*s.* 7¾*d.* each.

A	2359 × £1. =	£2359	0	0
B	„ × 14*s.* =	1651	6	0
C	„ × 6*d.* =	58	19	6
D	„ × 1½*d.* =	14	14	10½
E	„ × ¼*d.* =	2	9	1¾
		£4086	9	6¼

C is found as in Case VI.
D „ by dividing C by 4,
E „ „ D by 6.

Find the cost of 947 articles at 16*s.* 11¾*d.* each.

A	947 × 16*s.* =	£757	12	0
B	„ × 8*d.* =	31	11	4
C	„ × 3*d.* =	11	16	9
D	„ × ¾*d.* =	2	19	2¼
		£803	19	3¼

B and C are obtained as in Case VI.
D is one-fourth of C.

EXERCISE XLI.

		£.	*s.*	*d.*
(1)	7000 @	5	10	0
(2)	394	1	18	0
(3)	1776	4	17	6
(4)	3458	2	13	4
(5)	43728	8	18	6
(6)	4639	1	17	3
(7)	529 @	4	8	7½
(8)	7641	3	7	11¾
(9)	82465	0	13	9¼
(10)	8762	10	1	1¼
(11)	385	5	7	6¾
(12)	1044	3	19	10½

			£.	s.	d.				£.	s.	d.
(13)	157	@	6	18	$9\frac{3}{4}$	(20)	854	@	0	5	3
(14)	7235		10	16	$7\frac{1}{4}$	(21)	720		11	10	$7\frac{1}{4}$
(15)	586		4	4	$4\frac{1}{2}$	(22)	37		10	6	$8\frac{1}{2}$
(16)	725		0	12	$11\frac{1}{2}$	(23)	256		20	11	$4\frac{3}{4}$
(17)	678		0	9	$2\frac{1}{4}$	(24)	749		40	14	6
(18)	965		3	2	$5\frac{1}{2}$	(25)	365		0	15	$5\frac{1}{4}$
(19)	132		7	3	$8\frac{3}{4}$	(26)	1870		8	14	$9\frac{1}{2}$

(27) Taking a year to be 365 days, find the wages for a year, if the daily wages be 1*s.*, 4*s.* 6*d.*, 7*s.* 4*d.*, £1. 2*s.* 6*d.*, £3. 4*s.* 10*d.*

CASE VIII. Find the cost of $2037\frac{1}{2}$ articles at £4. 11*s.* 7*d.* each.

2037 × £4. =	£8148	0	0
,, × 10*s.* =	1018	10	0
,, × 1*s.* =	101	17	0
,, × 6*d.* =	50	18	6
,, × 1*d.* =	8	9	9
$\frac{1}{2}$ of £4. 11*s.* 7*d.* =	2	5	$9\frac{1}{2}$
	£9330	1	$0\frac{1}{2}$

$2675\frac{4}{9}$ articles at £1. 18*s.* $3\frac{1}{2}d.$

2675 × £1. =	£2675	0	0	
,, × 18*s.* =	2407	10	0	
,, × 3*d.* =	33	8	9	18
,, × $\frac{1}{2}d.$ =	5	11	$5\frac{1}{2}$	9
$\frac{1}{9}$ of £1. 18*s.* $3\frac{1}{2}d.$ =	0	4	$3\frac{1}{18}$	1
$\frac{3}{9}$,, =	0	12	$9\frac{1}{6}$	3
	£5122	7	$2\frac{13}{18}$	13

Similarly, if the fraction of the article were $\frac{4}{5}$, $\frac{8}{13}$, &c., we may first find the cost of $\frac{1}{5}$, $\frac{1}{13}$, &c., and then by multiplication the cost of the remaining $\frac{3}{5}$, $\frac{7}{13}$, &c.

EXERCISE XLII.

(1) $426\frac{1}{2}$ × £8. 4*s.* $3\frac{1}{2}d.$

(2) $712\frac{2}{3}$ × £5. 7*s.* 4*d.*

(3) $28\frac{15}{16}$ × £20. 4*s.* 4*d.*

(4) $537\frac{7}{11}$ × £33. 4*s.* 9*d.*

(5) $821\frac{5}{8}$ × £7. 10*s.* $3\frac{3}{4}d.$

(6) $246\frac{7}{20}$ × £14. 4*s.* $4\frac{1}{4}d.$

CASE IX.* Find the cost of 9 cwt., 3 qrs., 18 lbs. at £7. 10*s.* 8*d.* per cwt.

A	9 cwt. at £7. 10*s.* 8*d.*		=	£67	16	0
B	2 qrs.	,,	=	3	15	4
C	1 qr.	,,	=	1	17	8
D	14 lbs.	,,	=	0	18	10
E	4 lbs.	,,	=	0	5	$4\frac{4}{7}$
				£74	13	$2\frac{4}{7}$

B. 2 qrs. = $\frac{1}{2}$ of 1 cwt. Find $\frac{1}{2}$ of £7. 10*s.* 8*d.*
C. 1 qr. = $\frac{1}{2}$ of 2 qrs. ,, $\frac{1}{2}$ of B.
D. 14 lbs. = $\frac{1}{2}$ of 1 qr. ,, $\frac{1}{2}$ of C.
E. 4 lbs. = $\frac{1}{7}$ of 1 qr. ,, $\frac{1}{7}$ of C.

Find the value of 7 oz., 14 dwts., 11 grs. of gold, at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.

							320
7 oz. at £3. 17*s.* $10\frac{1}{2}$*d.*		=	£27	5	$1\frac{1}{2}$	—	160
10 dwts.	,,	=	1	18	$11\frac{1}{4}$	—	80
4 ,,	,,	=	0	15	$6\frac{9}{10}$	32	288
8 grs.	,,	=	0	1	$3\frac{23}{40}$	8	184
2 ,,	,,	=	0	0	$3\frac{143}{160}$	2	286
1 gr.	,,	=	0	0	$1\frac{303}{320}$	—	303
			£30	1	$5\frac{21}{320}$		$\frac{1301}{320} = 4\frac{21}{320}$.

EXERCISE XLIII.

(1) 7 tons, 5 cwt., 1 qr., 21 lbs. at £10. 4*s.* per ton.
(2) 11 cwt., 3 qrs., 14 lbs. at £8. 10*s.* per ton.
(3) 4 cwt., 1 qr., 17 lbs. at £3. 3*s.* per cwt.
(4) 4 oz., 11 dwt., 17 grs. at £5. 10*s.* per oz.
(5) 14 miles, 2 fur., 143 yds. at £42 per mile.
(6) 3 qrs., 19 lbs. at £7. 8*s.* 4*d.* per cwt.
(7) 4 hours, 17 min., 17 sec. at 45 miles per hour.
(8) 11 tons, 11 cwt., 23 lbs. at £3. 19*s.* per ton.
(9) 4 oz., 7 dwts., 11 grs. at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.
(10) 23 lbs., 14 oz. av. at £18. 10*s.* per cwt.

* Students who intend to pursue the study of Arithmetic into Part III. are advised not to waste much time over this barbarous rule, which we give reluctantly in deference to use and wont.

CASE X. Find the cost of 149 tons, 13 cwt., 3 qrs., 10 lbs., at £43. 8*s.* 4*d.* per ton.

		£	s.	d.		112
149 × £43	=	£6407	0	0		
,, × 8*s.*	=	59	12	0		
,, × 4*d.*	=	2	9	8		
10 cwt. at £43. 8*s.* 4*d.* per ton		21	14	2		
2 ,, ,,	=	4	6	10		
1 ,, ,,	=	2	3	5		
2 qrs. ,,	=	1	1	$8\frac{1}{2}$	—	56
1 qr. ,,	=	0	10	$10\frac{1}{4}$	—	28
7 lbs. ,,	=	0	2	$8\frac{9}{16}$	7	63
1 lb. ,,	=	0	0	$4\frac{73}{112}$	—	73
2 lbs. ,,	=	0	0	$9\frac{17}{56}$	2	34
		£6499	2	$6\frac{15}{56}$		$\frac{254}{112}=2\frac{30}{112}\Big\vert\frac{15}{56}$.

Find the dividend on £537. 8*s.* 10*d.* at 9*s.* $7\frac{3}{4}$*d.* in the £.

		£	s.	d.		480
Dividend of 20*s.* in the £	=	£537	8	10		
,, 5*s.* ,,	=	134	7	$2\frac{1}{2}$	—	240
,, 4*s.* ,,	=	107	9	$9\frac{1}{5}$	—	96
,, 6*d.* ,,	=	13	8	$8\frac{13}{20}$	24	312
,, $1\frac{1}{2}$*d.* ,,	=	3	7	$2\frac{13}{80}$	6	78
,, $\frac{1}{4}$*d.* ,,	=	0	11	$2\frac{173}{480}$	—	173
		£259	4	$0\frac{419}{480}$		$\frac{899}{480}=1\frac{419}{480}d.$

EXERCISE XLIV.

(1) 421 tons, 11 cwt. at £5. 12*s.* 6*d.* per ton.

(2) 75 tons, 9 cwt., 1 qr. at £12. 10*s.* 6*d.* per ton.

(3) 137 oz., 12 dwt. at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.

(4) 511 miles, 5 fur., 77 yds. at £417. 18*s.* 9*d.* per mile.

(5) 4*s.* $10\frac{1}{2}$*d.* in the £ on £5811. 17*s.* 6*d.*

(6) 71 days (of 12 hrs.), $9\frac{1}{2}$ hrs. at £7. 7*s.* a day.

(7) 111 yds., 1 ft., 1 in. at 5*s.* $6\frac{1}{2}$*d.* per yd.

(8) 5*s.* $4\frac{3}{4}$*d.* in the £ on £826. 14*s.* 10*d.*

(9) 37 cwt., 3 qrs., 24 lbs. at £10 12*s.* 8*d.* per cwt.

(10) 76 gals., 3 qts., 1 pt. at 14*s.* 9*d.* per gal.

(11) 23 years, 17 weeks, at £10. 10*s.* a year.

(12) Find the price of 14 ingots of gold, each weighing 3 lbs., 7 oz., 14 dwts., 21 grs. at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.

(13) 3 cwt., 2 qrs., 16 lbs., at £3. 7*s.* 8*d.* per cwt.

(14) 3 acres, 1 rood, 14 poles, at £125 per acre.

(15) 15 silver plates, each weighing 7 oz., 11 dwts., 6 grs. at 6*s.* 8*d.* per oz.

CHAPTER VII.

PROPORTION.

§ 1. Find a number that shall be the same fraction of 12 that 6 is of 8. 6 is $\frac{6}{8}$ or $\frac{3}{4}$ of 8, ∴ we have to find $\frac{3}{4}$ of 12. *Ans.* 9.

EXERCISE XLV.

What quantity is the same fraction

(1)	of 20	that 4	is of 5?
(2)	„ 35	„ 6	„ 15?
(3)	„ 35	„ 15	„ 6?
(4)	„ $2\frac{3}{4}$	„ $1\frac{1}{2}$	„ $1\frac{3}{4}$?
(5)	„ $10\frac{4}{5}$	„ $1\frac{4}{5}$	„ 2?
(6)	„ $10\frac{4}{5}$	„ $10\frac{4}{5}$	„ 11?
(7)	„ £2. 10*s.* 8*d.*	„ 3	„ 4?
(8)	„ £10. 10*s.*	„ 1 cwt., 3 qrs.	„ 2 cwt., 2 qrs.?
(9)	„ £8. 7*s.* 6*d.*	„ 2 yrs., 73 d.	„ $3\frac{1}{2}$ yrs.?
(10)	„ 1 oz. troy	„ 1*s.*	„ £1?
(11)	„ 1 oz. troy	„ 1*d.*	„ £1?
(12)	„ 1 lb. troy	„ 1*s.*	„ £1?
(13)	„ £1	„ 1 oz.	„ 1 lb. troy?
(14)	„ £1	„ 1 dwt.	„ 1 lb. troy?
(15)	„ £1	„ 1 qr.	1 ton?
(16)	„ £1	„ 1 fur.	„ 1 mile?
(17)	„ 2*s.* 6*d.*	„ 1 pole	„ 1 furlong?
(18)	„ £1. 1*s.*	„ $6\frac{2}{3}$ yds.	„ $3\frac{4}{5}$ furlongs?
(19)	„ $\frac{1}{4\frac{2}{7}}$	„ $3 \times 1\frac{2}{7}$	„ $6 \times 1\frac{2}{7}$?
(20)	„ The reciprocal of 10	„ 11	„ its reciprocal?
(21)	„ 1000 sheep	„ 3 men	„ 5 men?

§ 2. The fraction that one number is of another is called the RATIO of the first number to the second. In every ratio, therefore, there must be two quantities (or "terms") of which the first is called the *Antecedent* and the second is called the *Consequent*.

Of the ratio $\frac{3}{4}$, the numerator 3 is the antecedent, 4 the consequent. The ratio $\frac{3}{4}$ is commonly written $3:4$, and is read 3 *to* 4. We see that $3:4$ is one more interpretation of $\frac{3}{4}$, and we have the following sets of interchangeable terms. (Cf. p. 21.)

Numerator,	Denominator,	Fraction,
Dividend,	Divisor,	Quotient,
Antecedent,	Consequent,	Ratio.

[Note, however, that every ratio must be an abstract number (or fraction), and hence that the Antecedent and Consequent must be quantities of the same kind. Ratio, in fact, is Relative Magnitude, which can of course only subsist between quantities whose magnitudes can be compared.]

§ 3. Since antecedent, consequent and ratio are only different names for numerator, denominator and fraction, whatever holds true of fractions must also hold true of ratios. Thus, for example,

By multiplying or dividing both terms of a fraction by the same number the value of the fraction is unaltered (Ch. II. § 1). Similarly,

By multiplying or dividing both terms of a ratio by the same number the ratio is unaltered.

E.g.: since $\frac{4}{5}=\frac{8}{10}$, $4:5=8:10$.
Again: as $\frac{3}{4}>\frac{2}{3}$, so $3:4>2:3$.

EXERCISE P.

Which is the greater of each of the following ratios:

(1) $5:8$ or $6:9$.
(2) $7:10$ or $6:9$.
(3) $10:7$ or $9:6$.
(4) $8:9$ or $10:12$.
(5) $4:8$ or $8:14$.
(6) $4:8$ or $6:12$.
(7) $7\frac{1}{2}:10$ or $3:4$.
(8) $\frac{2}{3}:\frac{1}{2}$ or $\frac{2}{3}:\frac{5}{8}$.
(9) $\frac{2}{3}:\frac{1}{2}$ or $\frac{3}{8}:\frac{1}{2}$.
(10) $25:30$ or $15:18$.

§ 4. When two ratios are equal, the four terms are said to be in PROPORTION; thus the four numbers, 25, 30, 15, 18, are in proportion; since $25 : 30 = 15 : 18$. (Read: 25 *is to* 30 *as* 15 *is to* 18.)

In every proportion there must therefore be four terms, viz. two antecedents (25 and 15), and two consequents (30 and 18). The first and fourth, or outer terms, are called the Extremes; the second and third, or inner terms, are called the Means. It is also convenient to call the two extremes or the two means *similar* terms, and one extreme and one mean *dissimilar* terms. Thus in the above proportion 25 and 18 are similar, and so are 30 and 15; but 25 and 30, 25 and 15, 30 and 18, 15 and 18, are dissimilar.

Notice that since by a proportion we mean the equality of two ratios, a proportion is true only if the two ratios or fractions reduced to lowest terms prove to be identical; thus $\frac{25}{30} = \frac{5}{6}$, and $\frac{15}{18} = \frac{5}{6}$.*

§ 5. There is another mode of testing proportionality. To compare two fractions we reduce them to *some* common denominator, for which we may choose the product of the two denominators.

E.g. to compare $\frac{25}{30}$ with $\frac{15}{18}$, we may reduce them to the common denominator 18×30, and they become $\frac{25 \times 18}{30 \times 18}$ and $\frac{15 \times 30}{18 \times 30}$, and generally the numerators will be respectively the first numerator × the second denominator, and the second numerator × the first denominator. If these products are equal, the fractions are equal, and if not, not.

Translating this conclusion into the language of ratios: Two ratios are equal if the first antecedent × second consequent = second ante-

* Two fractions at lowest terms cannot be equal unless they are identical. To compare, for example, $\frac{5}{24}$ with $\frac{7}{36}$ we multiply each numerator by the number of times the other denominator contains the G.C.M. of the denominators, and these two multipliers are therefore prime to each other. In this case they are 3 and 2. The new numerator of the first fraction will then contain 3; but the new numerator of the second fraction cannot contain 3, for the old numerator did not contain 3, the fraction being at lowest terms, and the multiplier now introduced cannot contain 3, being prime to it. Hence to suppose the two fractions equal, involves the possibility of an integer divisible by 3 being equal to an integer prime to 3, which is absurd.

cedent × first consequent. In other words: Four quantities are in proportion if the product of the extremes is equal to the product of the means, and if not, not.

If any number be broken up into two pairs of factors (as $36 = 4 \times 9 = 3 \times 12$), these two pairs are the similar terms of a proportion ($4 : 3 = 12 : 9$); the product of the extremes being equal to the product of the means.

§ 6. The four numbers, 4, 3, 12, 9, can be arranged in eight different ways, in each of which the tests for proportionality will hold.

$$\begin{array}{ll} 4 : 3 = 12 : 9. & 12 : 9 = 4 : 3. \\ 4 : 12 = 3 : 9. & 12 : 4 = 9 : 3. \\ 3 : 4 = 9 : 12. & 9 : 12 = 3 : 4. \\ 3 : 9 = 4 : 12. & 9 : 3 = 12 : 4. \end{array}$$

There are sixteen other ways of arranging these four figures, but in no one of them does either test hold.

EXERCISE XLVI.

(1) Break up 60 into two pairs of factors, and write down the resulting eight proportional arrangements.

(2) Do the same with 15, 21, 25, 100, $2\frac{5}{8}$.

§ 7. The question of most frequent occurrence in Proportion is the following: Given any three terms to find the fourth. Putting the letter x for the missing term, the following four cases arise:

$$\begin{array}{ll} \text{i.} & x : 12 = 10 : 15. \\ \text{ii.} & 8 : x = 10 : 15. \\ \text{iii.} & 8 : 12 = x : 15. \\ \text{iv.} & 8 : 12 = 10 : x. \end{array}$$

(i.) translated means: A certain number is the same fraction of 12 that 10 is of 15. Find the number.

$$10 \text{ is } \tfrac{10}{15} \text{ of } 15.$$
$$\therefore x \text{ is } \tfrac{10}{15} \text{ of } 12. \qquad \textit{Ans.} \ \frac{10 \times 12}{15} = 8.$$

(ii.) means: 8 is the same fraction of a certain number that 10 is of 15. Find the number.

$$10 \text{ is } \tfrac{10}{15} \text{ of } 15.$$
$$\therefore 8 \text{ is } \tfrac{10}{15} \text{ of } x.$$
$$\text{or } x \text{ is } \tfrac{15}{10} \text{ of } 8. \quad (\text{Ch. III. § 6.}) \qquad \textit{Ans.}\ \frac{15 \times 8}{10} = 12.$$

(iii.) means: 8 is the same fraction of 12 that a certain number is of 15. Find the number.

$$8 \text{ is } \tfrac{8}{12} \text{ of } 12.$$
$$\therefore x \text{ is } \tfrac{8}{12} \text{ of } 15. \qquad \textit{Ans.}\ \frac{8 \times 15}{12} = 10.$$

(iv.) means: 8 is the same fraction of 12 that 10 is of a certain number. Find the number.

$$8 \text{ is } \tfrac{8}{12} \text{ of } 12.$$
$$\therefore 10 \text{ is } \tfrac{8}{12} \text{ of } x.$$
$$\text{or } x \text{ is } \tfrac{12}{8} \text{ of } 10. \qquad \textit{Ans.}\ \frac{12 \times 10}{8} = 15.$$

The same result might have been obtained more readily from the test given in § 5.

If the missing term be an extreme, the question is, What number multiplied by the other extreme gives the product of the means? *Ans.* The product of the means divided by the other extreme.

If the missing term be a mean, the question is, What number multiplied by the other mean gives the product of the extremes? *Ans.* The product of the extremes divided by the other mean. Or, in other words, The product of any two similar terms divided by one dissimilar term gives the other dissimilar term.

Thus if $a : b = c : d$, $a = \frac{b \times c}{d}$; $b = \frac{a \times d}{c}$; $c = \frac{a \times d}{b}$; $d = \frac{b \times c}{a}$.

Since "the product of any two similar terms divided by one dissimilar term gives the other dissimilar term," the two similar terms to be multiplied will give the numerator of a fraction, and the given dissimilar term the denominator. Hence the dissimilar term

may be cancelled against *either* of the two similar terms, or if convenient the dissimilar term and *either* of the similar terms may be multiplied by the same number.

Thus if $8 : 10 = 6 : x$, then $4 : 5 = 6 : x$, and $2 : 5 = 3 : x$;
also $3 \times 8 : 3 \times 10 = 6 : x$, and $5 \times 8 : 10 = 5 \times 6 : x$,
and $3 \times 5 \times 8 : 3 \times 10 = 5 \times 6 : x$, &c.

Generally: If $a : b = c : d$, $m \times a : m \times b = c : d$, and $m \times a : b = m \times c : d$, &c., where m represents any multiplier, integral or fractional. In other words, "Any two dissimilar terms may be multiplied or divided by the same number without disturbing the proportion."

Exercise XLVII.

I. State the following proportion in the eight different ways and find the value of x from each way. $x : 10 = 12 : 15$.

II. Find the value of x from the following:

(1) $x : 2 = 3 : 4$.

(2) $2 : x = 3 : 4$.

(3) $2\frac{1}{2} : 3\frac{1}{2} = x : 4\frac{1}{2}$.

(4) $5\frac{1}{4} : 1 = 3 : x$.

(5) $4 : 5 = 5 : x$.

(6) $\dfrac{1}{2\frac{1}{3}} : x = \dfrac{1}{3\frac{1}{6}} : \dfrac{1}{2\frac{1}{2} + 3\frac{1}{3}}$.

(7) $\dfrac{1}{2 + \dfrac{1}{3\frac{1}{4}}} : \frac{13}{30} = x : 1$.

(8) $x : \dfrac{1}{4 + \dfrac{1}{5 + \dfrac{1}{6}}} = \dfrac{1}{6 + \dfrac{1}{5 + \dfrac{1}{4}}} : 21\frac{7}{10}$.

III. Place x in the first, second, third and fourth terms successively, and find its value in each case, the other three terms being:

(1) 12, 15, 20.

(2) 1, 2, 3.

(3) $3\frac{1}{3}$, $3\frac{3}{4}$, $2\frac{6}{7}$.

(4) $(\frac{1}{4} - \frac{1}{9})$, $(\frac{1}{2} + \frac{1}{3})$, $(\frac{1}{2} - \frac{1}{3})$.

§ 8. Find x from $2\frac{4}{5} : 7\frac{1}{2} = 8\frac{3}{4} : x$;

$$\frac{14}{5} : \frac{15}{2} = \frac{35}{4} : x.$$

$$\begin{array}{r|l} \times 5 & 14 : 5 \times \frac{15}{2} = \frac{35}{4} : x \\ \times 2 & 2 \times 14 : 5 \times 15 = \frac{35}{4} : x \\ \times 4 & 4 \times 2 \times 14 : 5 \times 15 = 35 : x \\ \div 7 & 4 \times 2 \times 2 : 5 \times 15 = 5 : x \end{array}$$

$$x = \frac{5 \times 5 \times 15}{4 \times 2 \times 2} = \frac{375}{16} = 23\frac{7}{16}.$$

This example shews that any denominator may be transferred as a factor to a dissimilar term. Shortly, thus:

$$\begin{array}{lll} \cancel{2\tfrac{4}{5}} : & \cancel{7\tfrac{1}{2}} = & \cancel{8\tfrac{3}{4}} : x \\ \cancel{14} & 5 & \cancel{35} \\ 2 & 15 & 5 \\ 4 & & \\ 2 & & \end{array} \qquad x=\frac{5\times5\times15}{2\times4\times2}=23\tfrac{7}{16}.$$

Examine the proportion £5 : £8 = £7 : x. Here we can attach no sense to the product of the means. (Part I. Ch. V. § 17.) The question is: Of what sum of money is £7 the same fraction that £5 is of £8? The concrete quantity £5 is the same fraction of the concrete £8 that the abstract number 5 is of 8, therefore substituting abstract for concrete numbers, the rules previously given become intelligible.

Find x from £1. 0*s*. $7\tfrac{1}{2}$*d*. : £1. 13*s*. = £1. 8*s*. 4*d*. : x

$$\begin{array}{llll} & £1\tfrac{1}{32} & : £1\tfrac{13}{20} & = £1.\ 8s.\ 4d. : x \\ \therefore & \tfrac{33}{32} & : \ \tfrac{33}{20} & \\ & \cancel{33} & \ \ \cancel{32} & \\ & \cancel{20} & \ \ \cancel{33} & \\ & 5 & \ \ 8 & \end{array} \qquad x=\tfrac{8}{5} \text{ of } £1.\ 8s.\ 4d. = £2.\ 5s.\ 4d.$$

or,

$$\begin{array}{ll} \cancel{20\tfrac{5}{8}s.} & : \cancel{33s.} = £1.\ 8s.\ 4d. : x \\ \cancel{165} & \ 8 \\ \cancel{15} & \ \cancel{3} \\ 5 & \end{array} \qquad x=\tfrac{8}{5} \text{ of } £1.\ 8s.\ 4d. \text{ as before.}$$

or,

$$\begin{array}{ll} \cancel{495} \text{ halfpence} & : \cancel{792} \text{ halfpence} = £1.\ 8s.\ 4d. : x \\ \cancel{55} & \ \ \cancel{88} \\ 5 & \ \ 8 \end{array} \qquad x=\tfrac{8}{5} \text{ of } £1.\ 8s.\ 4d. \text{ as before.}$$

We see that the choice of the unit is immaterial. It is best to choose the largest which is readily obvious.

§ 9. Find the cost of 35 articles at £1. 18*s*. $10\tfrac{1}{2}$*d*. for 21 articles.

The beginner will find it convenient first to arrange the problem thus:

Statement: 21 articles cost £1. 18*s*. $10\tfrac{1}{2}$*d*.
Question: 35 „ £x.

It is clear that the ratio of the two numbers of articles must equal that of the two prices, or that 21 : 35 = £1. 18*s.* 10½*d.* : x. Hence 3 : 5, &c.; $\therefore x = \dfrac{5 \times \text{£}1.\ 18s.\ 10\frac{1}{2}d.}{3} = \text{£}3.\ 4s.\ 9\frac{1}{2}d.$

If 14 articles cost £3. 17*s.* 7*d.*, how many articles can be bought for £1. 10*s.* 5¾*d.*?

$$\begin{array}{lccc} \text{Art.} & \text{£.} & s. & d \\ 14 \text{ cost} & 3 & 17 & 7 \\ x \quad ,, & 1 & 10 & 5\frac{3}{4} \end{array}$$

$$14 : x = \text{£}3\ 17\ \ 7 : \text{£}1\ 10\ \ 5\tfrac{3}{4}.$$

It is, however, usual to make x the fourth term; thus:

$$\begin{array}{llll} \text{£}3.\ 17s.\ 7d. & : \text{£}1.\ 10s.\ 5\frac{3}{4}d. & = \cancel{14} & : x \\ \cancel{3724} \text{ farthings} & \cancel{1463} \text{ farthings} & \cancel{2} & \\ \cancel{532} & 209 & 1 & \\ \cancel{266} & & \dfrac{209 \times 1}{38} = \dfrac{209}{38} = 5\dfrac{19}{38}\Big| \dfrac{1}{2}. & \\ 38 & & & \end{array}$$

Ans. 5½ articles.

If the profits on £125. 14*s.* 9¼*d.* be £6. 15*s.*, what capital must I invest to get a profit of £168. 15*s.*?

$$\begin{array}{lllll} & \text{Capital.} & & \text{Profit.} & \\ \text{£}125\ 14 & 9\frac{1}{4} & \text{brings} & \text{£}6\ 15 & 0 \\ & x & ,, & 168\ 15 & 0 \end{array}$$

$$\begin{array}{lll} \text{£}6.\ 15s.\ 0d. & : \text{£}168.\ 15s. & = \text{£}125.\ 14s.\ 9\frac{1}{4}d. : x \\ \cancel{135} \text{ shillings} & \cancel{3375} \text{ shillings} & \\ \cancel{27} & \cancel{675} & \\ \cancel{3} & \cancel{75} & \\ 1 & 25 & \end{array}$$

£125. 14*s.* 9¼*d.* × 25 = £3143. 9*s.* 3¼*d.* *Ans.*

If 11¼ articles cost £2. 5*s.* 10*d.*, what will 19$\frac{11}{16}$ articles cost?

$$\begin{array}{llll} \text{Art.} & \text{£.} & s. & d. \\ 11\frac{1}{4} \text{ cost} & 2 & 5 & 10 \\ 19\frac{11}{16}\ ,, & & x & \end{array}$$

$$\begin{array}{lll} \cancel{11\frac{1}{4}} & : \cancel{19\frac{11}{16}} & = \text{£}2.\ 5s.\ 10d. : x \\ \cancel{45} & \cancel{4} & \\ \cancel{16} & \cancel{315} & \\ 4 & \cancel{63} & \dfrac{\text{£}2.\ 5s.\ 10d. \times 7}{4} = \text{£}4.\ 0s.\ 2\frac{1}{2}d.\ Ans. \\ \cancel{9} & 7 & \end{array}$$

Find the cost of 1 ton, 15 cwt., 3 qrs., 5 lbs., if 13 cwt., 0 qrs., 21 lbs. cost £4. 16*s.* 6½*d.*?

Tons.	cwt.	qrs.	lbs.		£.	*s.*	*d.*
0	13	0	21	cost	4	16	6½
1	15	3	5	,,		x	

$$\begin{array}{ccccccccccccccccc} \text{Cwt.} & \text{qrs.} & \text{lbs.} & & \text{Tons.} & \text{cwt.} & \text{qrs.} & \text{lbs.} & & \text{£.} & s. & d. & & \\ 13 & 0 & 21 & : & 1 & 15 & 3 & 5 & = & 4 & 16 & 6\tfrac{1}{2} & : & x \\ & 1477 \text{ lbs.} & & : & & 4009 \text{ lbs.} & & & & & & & & \end{array}$$

$$\frac{\text{£}4.\ 16s.\ 6\tfrac{1}{2}d. \times 4009}{1477} = \text{£}13.\ 2s.\ 0\tfrac{1}{2}d.\ \textit{Ans.}$$

How much spirits of wine can I get for 4*s.* 3¾*d.* at 23*s.* per gallon?

Gallon.	£.	*s.*	*d.*
1	1	3	0
x		4	3¾

$$\text{£}1.\ 3s. : 4s.\ 3\tfrac{3}{4}d. = 1 : x$$

$$\begin{array}{ccccccccccc} \cancel{23} & \times & \cancel{12} & \times & \cancel{4} & : & \cancel{207} & = & \cancel{4} \times \cancel{2} & : & x \\ 1 & & \cancel{6} & & & & \cancel{9} & & & & \\ & & 2 & & & & 3 & & & & \end{array}$$

$$\tfrac{3}{2} = 1\tfrac{1}{2} \text{ pints. } \textit{Ans.}$$

Here £1. 3*s.* is expressed as 23 × 12 × 4 farthings without multiplying out; 4*s.* 3¾*d.* are 207 farthings, and 1 gallon is 4 × 2 pints. It is best, where possible, merely to indicate operations, without performing them, as this facilitates cancelling. Thus here the one gallon of the third term is expressed as 4 × 2 pints, because the factors thus introduced cancel against factors of the first term.

Exercise XLVIII.

(1) Find the cost of 72 articles, if 40 cost £3. 12*s.* 8½*d.*

(2) If 15 sacks of potatoes cost £4. 4*s.*, what shall we pay for 50 sacks?

(3) What is the cost of 104 yards at £1. 11*s.* 6*d.* for 91 yards?

(4) What will be the weekly keep of 195 horses, if 117 horses can be kept for £46. 16*s.* 9*d.*?

(5) What will be the carriage for 161 miles at 18*s.* 6*d.* for 276 miles?

(6) How many articles can be bought for £15 if 10 cost £25?

(7) If 15 sacks of potatoes cost £4. 4*s.*, how many can be bought for £2. 12*s.* 6*d.*?

(8) How many yards can be bought for £33. 6*s.* 8*d.* at £100 for 300 yards?

(9) If 45 yards of trench are dug by 15 men, how many men would be required to dig in the same time 63 yards?

(10) If a railway ticket for 125 miles cost 7*s.* $9\frac{3}{4}d.$, how far ought I to be able to travel for £3. 2*s.* 6*d.*?

(11) Find the cost of 4 tons, 17 cwt., 3 qrs., if 11 cwt., 2 qrs. cost 19*s.* $10\frac{1}{2}d.$?

(12) Gold costs £3. 17*s.* $10\frac{1}{2}d.$ per oz. troy. Find the value of 1 lb. av.?

(13) How many oz. troy of gold shall I get for £103. 16*s.* 8*d.*?

(14) Find the wages for 2 years, 8 months, at £10. 15*s.* per annum.

(15) A and B divide between them a hogshead of claret (23 dozen), costing £34. 10*s.* A takes 150 bottles. How much has B to pay?

(16) If 3 qrs., 5 lbs. cost 3*s.* $8\frac{1}{2}d.$, what will $2\frac{1}{2}$ cwt. cost?

(17) How long will £368. 19*s.* 1*d.* keep me, if £96. 0*s.* 7*d.* suffices from March 17 to June 20?

(18) How much old brandy can I get for £1. 17*s.* 6*d.* at £3 per gallon?

(19) Taking the diameter of the earth at 7917 miles, and the highest mountain at 29,000 feet, by what fraction of an inch ought this mountain to be represented on a globe 1 yard in diameter?

(20) If 6 cwt., 1 qr., 21 lbs., cost £5. 9*s.* $5\frac{1}{4}d.$, what weight can be bought for £17. 17*s.*?

(21) $4\frac{7}{8}$ articles cost 9*s.* 10*d.* Find the cost of $10\frac{4}{5}$ articles.

(22) How many articles can be bought for 17*s.* $4\frac{1}{2}d.$, if $8\frac{4}{7}$ articles are bought for 2*s.* $2\frac{2}{5}d.$?

(23) A owns $\frac{5}{16}$ of a ship, B owns $\frac{5}{12}$ of it. If A's share is worth £2950, what is B's worth?

(24) If for £1700 I buy $\frac{9}{10}$ of a ship, what fraction of it can I buy for 1270 guineas?

(25) Find the fourth proportional to $4\frac{3}{8}$, $5\frac{2}{5}$, $7\frac{11}{12}$ [i.e. place x in the fourth term].

(26) The value of a fraction is $\frac{19}{20}$, the numerator is $33\frac{5}{8}$. What must the denominator be?

§ 10. Compare the following questions:

(1) If 8 articles cost £30, what will 12 cost?

(2) If 8 men do a piece of work in 30 days, how long will 12 men require?

In (1) more articles cost *more* money; in (2) more men require *less* time. Articles and their prices, horses and their provender, men and the work they can do, &c., are said to be in *direct* proportion, which means that twice as many articles cost twice as much money, three times as many articles cost three times as much, and so on, or twice as many horses require twice as much provender, &c. But men and the time required are in a different relation to each other; for twice as many men require half the time, three times the number of men require one-third of the time, and so on. Quantities thus related to each other are said to be in *inverse* proportion.

Direct.	Inverse.
(1) $\downarrow$ If 8 men earn £30, $\downarrow$	(2) $\uparrow$ If 8 men require 30 days, $\downarrow$
$\downarrow$ 12 men will earn £x. $\downarrow$	$\uparrow$ 12 „ x days. $\downarrow$

In (1) the number 30 is to be altered in the ratio of 8 : 12.
In (2) „ „ 12 : 8.

Hence the resulting proportions are:

(1) $8 : 12 = 30 : x$. *Ans.* £45. (2) $12 : 8 = 30 : x$. *Ans.* 20 days.

Before "stating" a proportion, we must always determine whether the problem is one of direct or of inverse proportion. The beginner will find it convenient to indicate his conclusion by arrows as above.

EXERCISE XLIX.

(1) If 8 men can mow a field in 5 days, how many men will mow it in 2 days?

(2) If 8 men can mow a field in 5 days, how many days will 10 men require?

(3) If we require $75\frac{3}{4}$ yds. of carpet $\frac{3}{4}$ yds. wide, to cover a room, how many yards of carpet $1\frac{1}{8}$ yds. wide, will be required?

(4) If £483 gain £27. 15*s.* interest in 1 year, how much capital will make the same profit in $7\frac{1}{2}$ months?

(5) If with a given sum of money I can buy 112 dozen at 7*s.* $4\frac{1}{2}$*d.* per dozen, how many articles can I buy at 10*s.* 6*d.* per dozen?

(6) If the wages of an Austrian workman be 1*s.* 8*d.* a day, and of a Lancashire workman 4*s.*, how many of the former can be hired for the wages of 375 of the latter?

(7) What sum of money will at $3\frac{1}{2}$ per cent. yield the same interest that 400 guineas yield at 5 per cent.?

(8) The capital of a company is raised by the issue of 1750 shares at £45 each. What would be the value of each share, if 3000 shares had been issued to raise the same amount?

(9) If 6352 stones, 3 ft. long, are required for a certain wall, how many stones, 2 ft. long, will be wanted?

(10) What length of land must be cut off from a piece $13\frac{1}{2}$ poles wide, to contain an area equal to a field 88 yds. long and 55 yds. broad?

(11) If $6\frac{2}{3}$ bushels last $1\frac{7}{8}$ days, how many days will $14\frac{2}{9}$ bushels last?

(12) If the daily allowance is $6\frac{2}{3}$ bushels, the store will last $1\frac{7}{8}$ days. How long will the store last with a daily allowance of $14\frac{2}{9}$ bushels?

§ 11. CHAIN RULE.

All questions of proportion can be stated in the shape of a fraction. It is convenient to draw the line perpendicularly. Find the cost of 45 articles if 20 cost £6. 6*s*. This question stated thus:

$$\begin{array}{r|l} \pounds x & 45 \text{ art.} \\ 20 \text{ art.} & \pounds 6.\ 6s. \end{array}$$

is read: "How many pounds for 45 articles if 20 articles cost £6. 6*s.*?" 45 × £6. 6*s*. is the numerator and 20 the denominator of the fraction sought. Cancelling, we obtain, 9 × £1. 11*s*. 6*d*. = £14. 3*s*. 6*d*. *Ans*. £14. 3*s*. 6*d*.

$$\begin{array}{r|l} x & \cancel{45} \\ \cancel{20} & \cancel{6}\ \ \cancel{6}\ \ \cancel{0} \\ \cancel{4} & 9 \\ 1 & 1\ \ 11\ \ 6 \end{array}$$

The "chain" is correct for *direct* proportion (1) if each line begins with the denomination with which the previous line ended, (2) if the chain finishes with the same denomination with which it began, and (3) if the original sense of the question is preserved when read off as above.

This rule is applicable with advantage to such questions as the following: How many pounds of tea must be given for 28 lbs. of rice, if 4 lbs. of sugar are worth 1 lb. of coffee, 15 lbs. of sugar are worth 14 lbs. of rice, and 30 lbs. of coffee are worth 7 lbs. of tea?

Denominator.		*Numerator.*
tea	x	28 rice
rice	14	15 sugar
sugar	4	1 coffee
coffee	30	7 tea

This chain is stated correctly; for the second and third terms, the fourth and fifth terms, the sixth and seventh terms are respectively of like denomination, and so also are the first and last. Moreover, when read off, the statement preserves the original sense. The problem might be broken up into a number of separate questions of direct proportion. Thus:

How much sugar for 28 rice if 14 rice = 15 sugar? *Ans.* $\frac{28 \times 15}{14}$ sugar.

How much coffee for $\frac{28 \times 15}{14}$ sugar if 4 sugar = 1 coffee? *Ans.* $\frac{28 \times 15 \times 1}{14 \times 4}$ coffee.

How much tea for $\frac{28 \times 15 \times 1}{14 \times 4}$ coffee if 30 coffee = 7 tea? *Ans.* $\frac{28 \times 15 \times 1 \times 7}{14 \times 4 \times 30}$ tea, which is the fraction obtained above in one statement.

x	$\cancel{28}$
$\cancel{14}$	$\cancel{15}$
4	1
$\cancel{30}$	7
$\cancel{2}$	$\cancel{2}$

$\frac{7}{4} = 1\frac{3}{4}$. *Ans.* $1\frac{3}{4}$ lbs. tea.

Questions of this kind occur very rarely, but problems of "arbitration of exchange" are of daily occurrence, and are solved most readily by Chain Rule.

How many pounds sterling must be paid for 32830 francs, if 35 francs are worth 36 roubles; $4\frac{1}{5}$ Austrian florins $= 2\frac{7}{10}$ Prussian thalers, 12 roubles = 5 Austrian florins, and £1 $= 6\frac{7}{10}$ thalers.

£	x	$\cancel{32830}$ fr.
fr.	$\cancel{35}$	$\cancel{36}$ r.
r.	$\cancel{12}$	5 fl.
fl.	$\cancel{4\frac{1}{5}}$	$\cancel{2\frac{7}{10}}$ th.
th.	$\cancel{6\frac{7}{10}}$	1 £
	$\cancel{21}$	$\cancel{5}$
	$\cancel{10}$	27
	$\cancel{67}$	$\cancel{10}$
	$\cancel{7}$	$\cancel{3}$
	$\cancel{7}$	$\cancel{4690}$
		$\cancel{670}$
		10

$5 \times 27 \times 10 = 1350$.
Ans. £1350.

Exercise L.

(1) Find the cost of 72 yds. at the rate of £5. 12*s.* 6*d.* for 10 yds.

(2) If $7\frac{1}{4}$ cwt. cost £26. 10*s.* 4*d.*, what will $43\frac{1}{2}$ cwt. cost?

(3) If 3 paces are equal to 2 yds., how many paces will there be in $106\frac{2}{3}$ yds.?

(4) If 5 balls of lead weigh as much as 9 of iron, and 7 of marble as much as 3 of iron, how many balls of marble will equal in weight 35 balls of lead?

(5) If 1 dollar is worth 4*s.* $1\frac{1}{2}$*d.* and also 5 fr., 17 centimes, find the value of a franc in sterling money (1 fr. = 100 centimes).

(6) How many francs are worth £1, if $49\frac{1}{2}d.$ buy one dollar, and 100 dollars fetch 517 fr.?

(7) Find the value of the wool from 14,000,000 sheep, at £8. 16s. per cwt., if 11 sheep yield 25 lbs. of wool.

(8) How many francs for £1, if 35 Flemish shillings = £1, 480 rees are worth 3 francs and 400 rees are worth $3\frac{1}{2}$ Flemish shillings?

(9) If the rent of land in France be 140 francs per hectare, calculate the rate per acre in English money, 25 francs being equal to £1 and 100 hectares to 247 acres.

(10) If 1 metre $= 39\frac{37}{100}$ in. and £1 $= 25\frac{3}{8}$ fr., what is the cost in English money of 1 yd. at $1\frac{9}{20}$ fr. per metre?

(11) If $11\frac{11}{20}$ Dutch florins are given for $24\frac{9}{10}$ francs, 385 florins for 442 marks Hambro' and $68\frac{1}{4}$ marks for 32 Russian roubles, how much English money should be given for 2010 roubles at $25\frac{1}{8}$ fr. per £1?

(12) If 3 lbs. of tea are worth 4 lbs. of coffee, and 6 lbs. of coffee are worth 20 lbs. of sugar, how many lbs. of sugar can be had for 9 lbs. of tea?

(13) The day's journey in Turkey being 10 hours of $4\frac{1}{2}$ English miles each, and 11 English miles being equal to 12 Roman miles, how many Roman miles are there in 13 days' journey in Turkey?

(14) If £3 = 20 thalers, 25 gulden = 62 francs, 25 thalers = 93 francs; find how many gulden are equal to £1 sterling.

(15) If £1 $= 25\frac{1}{2}$ francs, $9\frac{1}{3}$ florins = 20 francs, how many florins for £1?

(16) How much capillaire must be added to 580 gallons of dry gin, if to 100 gals. of gin is put 45 lbs. of sugar, and 1 gallon of capillaire has the sweetening power of 13 lbs. of sugar?

(17) Find the cost of 30 pieces of lead, each weighing 1 cwt., 12 lbs., at 16s. 4d. per cwt.

Exercise LI.

(1) A person takes 2 paces to walk 3 yds. How many yds. will he get over in 250 paces?

(2) How many paces will he take in a mile?

(3) The clothing of a regiment of 750 soldiers amounts to £2831. 5*s*. What will the clothing of 3500 men cost?

(4) A bankrupt owes £1954, his assets are £840. 12*s*. $6\frac{1}{2}d$. What will a creditor for £153 recover?

(5) The circumference of the earth at the equator is 24,900 miles. At what rate per hour is a person there carried round, one whole rotation being made in 23 h., 56 min.?

(6) If the 6*d*. loaf weighs 4 lbs. when flour costs £3. 5*s*. a sack, what ought it to weigh if flour cost £2. 15*s*. a sack?

(7) If the 4 lb. loaf cost 6*d*. when flour is £3. 5*s*. a sack, what ought it to cost when flour is £2. 15*s*. a sack?

(8) If the 6*d*. loaf weighs 4 lbs., what ought the $8\frac{1}{2}d$. loaf to weigh?

(9) If in a picture a tree 33 feet high is represented by a drawing $1\frac{1}{2}$ inches high, what should represent the height of a house 45 feet high?

(10) How high must a shrub be which is represented in the picture by $\frac{3}{7}$ inch?

(11) If a country 630 miles long is represented in a raised map by a length of $5\frac{1}{2}$ feet, how high ought a mountain of 15,750 feet to be represented on the map?

(12) If $1\frac{1}{2}$ inches represent the distance of the moon from the earth, how far off should the sun be placed, the actual distances being taken as 238,793 and 95,517,200 miles respectively?

(13) If $1\frac{1}{2}$ inches represent the distance of the sun from the earth, how far off should the nearest fixed star be placed, its distance being taken as 20,185,649,876,000 miles?

(14) If light takes 8 min., 13 secs. to travel from the sun to the earth, how long will it take from the moon to the earth, and how long from the star?

(15) The area of a certain garden is $1\frac{1}{2}$ acres, the width being 425 ft. What will its area be if the width be made 510 ft.?

(16) A certain garden is 440 feet long and 100 ft. broad. What would be the breadth of a garden of the same size whose length was 363 ft.?

(17) Find the cost of 7 tons, 12 cwt., 3 qrs., at £1. 10*s.* 10*d.* for 3 cwt., 1 qr.

(18) Find the cost of a tankard, weighing 1 lb., 7 oz., 14 dwts., at £1. 14*s.* 10*d.* for $5\frac{1}{2}$ oz.

(19) If the carriage of 15 tons, 15 cwt., cost £1. 12*s.* 6*d.*, what will be the charge for carrying 3 tons, 17 cwt.?

(20) If 15*s.* 9*d.* pays the carriage on 2 tons, 17 cwt., what weight can be carried for a guinea?

(21) If goods are carried 45 miles for 7*s.* 6*d.*, how far ought they to be carried for 10*s.* 6*d.*?

(22) If for a certain payment 15 tons, 10 cwt. are carried over 100 miles, how far ought 4 tons, 13 cwt. to be carried?

(23) If for a certain payment 10 tons, 9 cwt. are carried 150 miles, what weight ought to be carried over 200 miles?

(24) If from every £92. 10*s.* of capital I obtain £3 income, how much shall I obtain from £740?

(25) A bankrupt pays 9*s.* $4\frac{1}{2}$*d.* in the £. What will be lost on a debt of £324. 13*s.* 4*d.*?

(26) My gross income is £730. 15*s.* What will be left me after paying 5*d.* in the £ income-tax?

(27) Find a fourth proportional to:

a. 6, 9, 10.
b. 7, 8, 9.
c. $2\frac{1}{2}$, $3\frac{1}{3}$, $1\frac{7}{8}$.
d. $\frac{1}{7}$ of $\frac{1}{3}$, $\frac{1}{4}$ of $\frac{1}{5}$, $\frac{1}{7}$ of $\frac{1}{8}$.
e. $(\frac{2}{3}+\frac{1}{2})$, $(\frac{2}{3}-\frac{1}{2})$, $\frac{2}{3}$ of $\frac{1}{2}$.
f. $\frac{1}{2\frac{1}{2}}$, $\frac{1}{3\frac{1}{3}}$, $\frac{1}{4\frac{1}{2}}$.

(28) Find the cost of 2 tons, 7 cwt., 3 qrs., at £5. 18*s.* for 15 cwt., 1 qr.

(29) Find the cost of 73 yds., 1 qr., 3 nls., at £39. 11*s.* 8*d.* for 1000 yds.

(30) If $1\frac{7}{8}$ articles cost £$5\frac{1}{16}$, how much will $2\frac{13}{16}$ articles cost?

(31) If £$\frac{2}{7}$ buy $\frac{5}{7}$ of 1 article, what will £$2\frac{2}{5}$ buy?

(32) A train goes from London to Bristol in $3\frac{1}{3}$ hours, travelling at the rate of 3 miles in 5 minutes. (1) How far is it to Bristol?

(2) How long would it take to Exeter (194 miles) at the same rate? (3) How long will it take to Bristol and to Exeter if the rate be increased to 40 miles an hour?

(33) If 770 gallons of creosote have the heating power of $8\frac{3}{4}$ tons of coal, how many gallons a day would be required for a steamer which consumes 50 tons of coal daily?

(34) Find the cost of 1 lb. avoirdupois of gold, at £3. 17*s.* $10\frac{1}{2}$*d.* per oz. troy.

(35) Find the weight of 4361 sovereigns in av. weight at the same rate.

(36) 10 cubic inches of gold weigh as much as 193 cubic inches of water. What is the size of a nugget weighing as much as a cubic foot of water?

(37) If $4\frac{1}{2}$ tons of coal fill a cellar 9 ft. long, 5 ft. broad, 5 ft. high, what space will be required for the coal of a steamer carrying 3 weeks' consumption, at 20 tons per day?

(38) The space between the freezing and the boiling points of water is divided into 180, 80, 100 degrees respectively on Fahrenheit's, Réaumur's and the Centigrade thermometers. How many degrees of the second and third are equivalent to 18, 27, 45 and 63 degrees on the first?

(39) In Fahrenheit's thermometer freezing point is marked 32^0; on the others, zero; so that 32^0 F $= 0^0$ C $= 0^0$ R. Translate the following readings into readings of the other two: 41^0 F; 8^0 R; 40^0 C; 40^0 R; 90^0 C; 86^0 F; 0^0 C; 100^0 C; 100^0 F; 50^0 C; 50^0 F; 50^0 R.

(40) 90 degrees are 100 grades. How many grades are equal to $65\frac{3}{8}$ degrees, and how many degrees are equal to $65\frac{3}{8}$ grades?

(41) If 600 men can dig a cutting 750 yards long in 23 days, how long would 460 men take?

(42) How long would the 600 men take if the cutting were 800 yards long?

(43) What part of the 800 yards would $\frac{4}{5}$ of the number of men do?

(44) How long would $\frac{4}{5}$ of the men take to do the 800 yards?

(45) How many men would it take to do the 750 yards in $\frac{4}{5}$ of the time?

(46) If 24 men dig a ditch in 3 days, how long would they take to dig a ditch half as long again, half as deep again, and half as broad again?

(47) How many men would it take to dig the second ditch in the same time?

(48) What will be my new expenditure, supposing it to have been originally 300 guineas, if I alter it in the ratio of 7 to 12?

(49) If brickwork, 84 ft. high, $73\frac{1}{2}$ ft. long, cost £700, how must I reduce the thickness so that it may cost only £600? What reduction of height would have given me the same result? What would be the cost if both height and length were thus reduced?

(50) If 3 cwt., 2 qrs., 12 lbs. cost £9, what is the price of 6 lbs.?

(51) A bankrupt's debts amount to £4586. 8*s.*, and his effects to 3822 guineas. How much will a creditor receive on a debt of £700?

(52) A railway train travels $\frac{1}{4}$ of a mile in 18 seconds. How many miles an hour does it travel at this rate?

(53) If $35\frac{1}{2}$ lbs. of sugar cost £1. 2*s.* $2\frac{1}{4}$*d.*, how much will 2 cwt., 1 qr. 23 lbs. cost?

(54) If £100 put out to interest for 9 months becomes £103, what sum would amount to £193. 2*s.* 6*d.* in the same time?

(55) If $\frac{1}{143}$ of $3\frac{2}{3}$ of $\frac{7}{9}$ of $5\frac{1}{5}$ of 22 lbs. of sugar cost $8\frac{1}{4}$*d.*, how much will 1 ton, 11 cwt., 3 qrs. cost?

(56) If $\frac{3}{11}$ of $\frac{3}{2\frac{1}{2}}$ of $\frac{5}{27}$ of a ship cost £3710, what part of the ship can be bought for £100?

(57) If a garrison of 1500 men have provisions for 13 months, how long will their provisions last if it be increased by 700 men?

(58) If a man's step be 2 ft., 4 in., and a horse's 2 ft., 9 in., how many steps of the horse are equal to 108 steps of the man?

(59) If 432 and 750 be two among several factors of a number, what must I substitute for the latter if the former be altered to 540, in order that the final product may remain the same?

(60) A coat requiring $2\frac{3}{4}$ yards of cloth, $1\frac{1}{4}$ yds. wide, is lined with silk 39 in. wide. How many yards of silk will be required?

(61) If the quotient be altered from 730 to 219, what must the dividend be, which was before 1,000,000, so that the divisor may be unaltered?

§ 12. COMPOUND PROPORTION.

If 20 men earn £15 in 4 days, how many men will earn £24 in 6 days?

Here the result is affected not only by the alteration of the amount earned, but also by that of the number of days. The problem may be resolved into two separate proportions thus:

(1) If 20 men earn £15 in 4 days,
x " £24 in the same time.

$$15 : 24 = 20 : x. \qquad \textit{Ans.}\ \frac{20 \times 24}{15} \text{ men.}$$

(2) If $\frac{20 \times 24}{15}$ men earn £24 in 4 days,
x " the same in 6 days.

$$6 : 4 = \frac{20 \times 4}{15} : x.$$

$$\textit{Ans.}\ \frac{\overset{4}{\cancel{20}} \times \overset{4}{\cancel{24}} \times 4}{\underset{3}{\cancel{15}} \times \underset{1}{\cancel{6}}} = \frac{64}{3} = 21\tfrac{1}{3} \text{ men.*}$$

The same resulting fraction, $\frac{20 \times 24 \times 4}{15 \times 6}$, will evidently be obtained by the following method. Statement:

Men.	£.	Days.
20	15	4
x	24	6

$$\left.\begin{array}{r} 15 : 24 \\ 6 : 4 \end{array}\right\} = 20 : x$$

$$\textit{Ans.}\ \frac{20 \times 24 \times 4}{15 \times 6} \text{ as above.}$$

* The interpretation that must be given to such an answer as this is, that £24 will pay for 6 days the wages of 21 men, with enough to spare to pay the third part of the wages of another man.

Hence each pair of numbers of like denomination must be separately considered with reference to that pair in which the x occurs, marking each direct proportion by a downward, and each inverse proportion by an upward arrow.

If 9 men can in 10 days, of 12 hours each, dig a trench 45 yards long, $1\frac{1}{4}$ ft. wide, and $1\frac{7}{8}$ ft. deep, how many hours a day must 8 men work to dig a trench $4\frac{7}{8}$ yds. long, $2\frac{1}{2}$ ft. wide, and 5 ft. deep, in 13 days?

Men.	Days.	Hours.	Yds. long.	Ft. wide.	Ft. deep.
↑ 9	↑ 10	↓ 12	↓ 45	↓ $1\frac{1}{4}$	↓ $1\frac{7}{8}$
8	13	x	$4\frac{7}{8}$	$2\frac{1}{2}$	5

1st. Put a downward arrow to the pair containing x.

2nd. Consider the men: more men, less hours. Inverse. ∴ place to the men an upward arrow.

3rd. Consider the days: more days, less hours. Inverse. Arrow upwards.

4th. More length, more hours. Direct. Arrow downwards.

5th. More width, more hours. Direct. Arrow downwards.

6th. More depth, more hours. Direct. Arrow downwards.

Now "compound" the several proportion sums as follows:

$$\left.\begin{array}{rcl} \cancel{8} & : & \cancel{9} \\ \cancel{13} & : & \cancel{10} \\ \cancel{45} & : & \cancel{4\tfrac{7}{8}} \\ \cancel{1\tfrac{1}{4}} & : & \cancel{2\tfrac{1}{2}} \\ \cancel{1\tfrac{7}{8}} & : & \cancel{5} \end{array}\right\} = \underset{6}{\cancel{12}} : x$$

$$\begin{array}{cc} \cancel{8} & \cancel{39} \\ \cancel{2} & \cancel{5} \\ \cancel{5} & \cancel{4} \\ \cancel{15} & \cancel{8} \\ \cancel{5} & \cancel{3} \\ \cancel{2} & \cancel{5} \\ \cancel{5} & \end{array}$$

Ans. 6 hours.

The justification of this method of compounding will perhaps be rendered more evident to beginners by the use of the language of Fractions.

9 men require 12 hours, ∴ 8 men require $\frac{9}{8}$ of 12 hours.

Again, for 13 instead of 10 days, they require $\frac{10}{13}$ of $\frac{9}{8}$ of 12 hours.

For $4\frac{7}{8}$ yds. instead of 45 yds. length, they require $\frac{4\frac{7}{8}}{45}$ of $\frac{10}{18}$ of $\frac{9}{8}$ of 12 hours.

For $2\frac{1}{2}$ ft. instead of $1\frac{1}{4}$ ft. width, they require $\frac{2\frac{1}{2}}{1\frac{1}{4}}$ of $\frac{4\frac{7}{8}}{45}$ of $\frac{10}{18}$ of $\frac{9}{8}$ of 12 hours.

And lastly, for 5 ft. instead of $1\frac{7}{8}$ ft. depth, they require $\frac{5}{1\frac{7}{8}}$ of $\frac{2\frac{1}{2}}{1\frac{1}{4}}$ of $\frac{4\frac{7}{8}}{45}$ of $\frac{10}{18}$ of $\frac{9}{8}$ of 12 hours.

The only difference between this result and that previously obtained is, that the numerators here were in that case means, and the denominators extremes. Every pair may thus be looked upon as a fraction, and in each case we have to determine which is the numerator.

Any factor common to both members of the pair may be struck out before commencing operations.

If I travel 300 miles in 6 days of 8 hours each, in how many days of 10 hours shall I travel 450 miles, travelling half as fast again?

Miles.	Days.	Hours.	Speed.
$\cancel{300}$ 2	6	$\cancel{8}$ 4	1
$\cancel{450}$ 3	x	$\cancel{10}$ 5	$1\frac{1}{2}$

$$\left.\begin{array}{r} \cancel{2} : \cancel{3} \\ 5 : 4 \\ \cancel{1\frac{1}{2}} : 1 \\ \cancel{3}\ \ \ \cancel{2} \end{array}\right\} = 6 : x.$$

$$\frac{6 \times 4}{5} = \frac{24}{5} = 4\tfrac{4}{5}.$$

Ans. $4\frac{4}{5}$ days.

Exercise LII.

(1) If £240 be the wages of 6 men for 21 weeks, what will 28 men earn in 23 weeks?

(2) How many men can I employ for 7 months for £453. 12*s.*, if the wages of 50 men for 12 months is £1080?

(3) If I pay £$1\frac{3}{4}$ to 13 men for $3\frac{3}{4}$ days, what will be the wages of 30 men for $10\frac{2}{5}$ days?

(4) If 36 persons consume 240 pecks of corn in 30 days, how long will 100 pecks last 90 persons?

(5) If 15 pumps, working 8 hours a day, can raise 1260 tons of water in 7 days, how many pumps, working 12 hours a day, will be required to raise 7560 tons of water in 14 days?

(6) If 24,000 yds. of cotton cloth, $1\frac{3}{4}$ yds. wide, be worth £400 when raw cotton is at $4\frac{1}{2}d.$ per lb., what is the value of 12,600 yds. of cotton cloth, $1\frac{1}{4}$ yds. wide, when raw cotton is at $9\frac{9}{16}d.$ per lb., supposing the cost of manufacture to have risen in equal proportion?

(7) If the corn of 13 horses for 63 days cost £17. 6*s.* 8*d.* when corn is 4*s.* per bushel, how many horses will in 56 days consume corn to the value of £10. 13*s.* 4*d.* when corn is 4*s.* 6*d.* per bushel?

(8) If 48 pioneers in 5 days, of $12\frac{1}{2}$ hours each, can dig a trench $139\frac{3}{4}$ yds. long, $4\frac{1}{2}$ yds. wide and $2\frac{1}{2}$ yds. deep, how many hours per day must 360 pioneers work during 42 days to dig a trench $4910\frac{1}{16}$ yds. long, $4\frac{7}{8}$ yds. wide and $3\frac{1}{5}$ yds. deep?

(9) If 6 men can perform a piece of work in 12 days of 10 hours each, how many men will perform a piece of work four times as large in a fifth of the time, if they work the same number of hours a day, supposing that 2 of the second set can do as much work in an hour as 3 of the first set?

(10) What is the interest on £100 for 1 year, if the interest on £1303. 6*s.* 8*d.* for 10 years is £488. 15*s.*?

(11) If a block of marble be worth £50, what will be the value of a block twice as long, 3 times as broad, and $\frac{1}{4}$ of the thickness?

(12) If a cubic block cost £50, what would be the price of a cubic block 3 times as long?

(13) If a block of marble weigh 50 cwt., what is the weight of a block of iron, the dimensions of which are to those of the former in the proportion of 2 : 3, 3 : 4, and 4 : 5 respectively, weights of equal blocks of iron and marble being in the proportion of 77 : 27.

§ 13. Proportional Parts.

Break up a given quantity into two parts whose ratio shall be that of the numbers 2 and 3. The two parts are $\frac{2}{5}$ and $\frac{3}{5}$ of the given quantity, for these added together give the whole quantity,

and they are in the required ratio. Similarly, to break up a quantity into three parts whose ratios to one another are those of the numbers 4, 5, 6 (which is expressed by $4 : 5 : 6$), break up the whole into $4 + 5 + 6 = 15$ parts, and take 4, 5 and 6 such parts severally. *Ans* $\frac{4}{15}$, $\frac{5}{15}$ and $\frac{6}{15}$, or $\frac{4}{15}$, $\frac{1}{3}$ and $\frac{2}{5}$ of the given quantity.

Generally: Divide the given quantity by the sum of the given numbers, and multiply the quotient by each number separately.

The ratios of two or more numbers are not altered if they are all multiplied or divided by the same number (Cf. § 6); thus $2 : 3 : 4 = 20 : 30 : 40$, and $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} = 15 : 10 : 6$ (each term being multiplied by 30). The general rule for simplifying ratios is to divide the given numbers by their G. C. M. if they are integers, or to multiply them all by the L. C. M. of their denominators if they are fractions. This will evidently reduce them to the simplest integral form.

Divide £10. 10*s.* into 4 parts, having the ratios $4 : 7 : 8 : 9$.

$4 + 7 + 8 + 9 = 28$. The fractions are $\frac{4}{28}$, $\frac{7}{28}$, $\frac{8}{28}$, $\frac{9}{28}$ of £10. 10*s.*

$\frac{1}{28}$ of £10. 10*s.*	=		0	7	6
$\frac{4}{28}$,,	=		1	10	0
$\frac{7}{28}$,,	=		2	12	6
$\frac{8}{28}$,,	=		3	0	0
$\frac{9}{28}$,,	=		3	7	6
			£10	10	0

Ans. £1. 10*s.*; £2. 12*s.* 6*d.*; £3; £3. 7*s.* 6*d.*

N.B. It is well always to verify the separate results by addition. They should of course give the original number to be divided.

If three persons invest £200, £350 and £750 respectively in the purchase of a property which yields an income of £109. 4*s.*, how should this be divided among them?

$200 : 350 : 750 = 4 : 7 : 15$. The shares should be $\frac{4}{26}$, $\frac{7}{26}$ and $\frac{15}{26}$ of £109. 4*s.*

$\frac{1}{26}$ of £109. 4*s.*	=	£4	4	0
$\frac{4}{26}$,,	=	16	16	0
$\frac{7}{26}$,,	=	29	8	0
$\frac{15}{26}$,,	=	63	0	0
		£109	4	0

Ans. £16. 16*s.*; £29. 8*s.*; £63.

If A invest £300 for 4 months, B £150 for 7 months, and C £200 for 9 months, how should the profit be divided?

If we take as a unit the investment of £1 for 1 month, A has invested 300×4 such units, B 150×7 units, and C 200×9 units.

$$\begin{aligned} &\cancel{300} \times 4 : \cancel{150} \times 7 : \cancel{200} \times 9 \\ = {} &\cancel{6} \times 4 : \cancel{3} \times 7 : 4 \times \cancel{9} \\ = {} &8 : 7 : 12, \end{aligned}$$

and the shares are $\frac{8}{27}$, $\frac{7}{27}$ and $\frac{12}{27}$ of the profit.

The sum of £15 was to be divided among the three head boys of a class in proportion to their marks. They obtained respectively $\frac{3}{4}$, $\frac{2}{3}$ and $\frac{1}{2}$ of the total number. What should each get?

$\frac{3}{4} : \frac{2}{3} : \frac{1}{2} = 9 : 8 : 6$, $\therefore$ $\frac{9}{23}$, $\frac{8}{23}$ and $\frac{6}{23}$ of £15 are the respective shares. *Ans.* £5. 17*s.* $4\frac{16}{23}$*d.*; £5. 4*s.* $4\frac{4}{23}$*d.*; £3. 18*s.* $3\frac{3}{23}$*d.*

Exercise LIII.

(1) Divide £3. 10*s.* into two parts which shall have the ratio of 5 : 7.

(2) Divide a guinea into three parts in the ratio of 2 : 3 : 4.

(3) Divide a guinea into 6 parts which shall have the ratios of the first 6 natural numbers.

(4) Two partners engage in business with capitals of £7000 and £9000 respectively, the profits amounting to £2400 a year. What should each receive?

(5) What is the value of the gold in a chain weighing 3 oz., 4 dwt. troy, supposing it to be 18 carats fine (i.e. 18 parts pure gold out of 24) at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.

(6) If two partners engage in business, investing respectively £482. 1*s.* 8*d.* and £630. 8*s.* 4*d.*, what should each have of a profit of £51. 17*s.* 6*d.*?

(7) Four persons, A, B, C, D, rent a pasture for £57; A put in 8 cattle, B 9, C 10, and D 11. How much should each pay for his share?

(8) A tax of £489. 17*s.* is to be raised from 3 towns, the populations of which are respectively 2500, 3000 and 4200. How much should each town pay?

(9) 10 gallons of alcohol are mixed with 14 gallons of water, What weight of alcohol is there in every lb. weight of the mixture, the weights of equal measures of alcohol and water being in the ratio 4 : 5.

(10) Copper is $8\frac{9}{10}$, tin $7\frac{3}{10}$ times as heavy as water. If 20 cubic inches of tin be mixed with 30 inches of copper, how many times as much as its own bulk of water will the mixture weigh?

(11) Four merchants, A, B, C, D, trade together. A's capital of £800 was in trade 8 months; B's of £700, 12 months; C's of £400, 6 months; and D's of £135, 4 months. What share of the profit should each receive?

(12) If 200 oz. of gold, 18 carats fine, are mixed with 128 oz., 15 carats fine, what is the weight of gold in the mixture?

(13) What is the fineness?

(14) Gunpowder contains $\frac{3}{4}$ of its weight of saltpetre; saltpetre is composed of 39 parts by weight of potassium, 14 of nitrogen and 48 of oxygen. How many lbs. of potassium are there in 909 lbs. of gunpowder?

(15) Divide £250 among A, B, C and D, so that A's share shall be to B's as 4 to 5, B's to C's as 5 to 6, C's to D's as 6 to 7.

(16) Divide £1000 among A, B, C and D, so that A's share shall be to B's as $\frac{1}{2}$ to $\frac{1}{3}$, B's to C's as $\frac{1}{3}$ to $\frac{1}{4}$, C's to D's as $\frac{1}{4}$ to $\frac{1}{5}$.

(17) Divide £31. 12*s.* 6*d.* among A, B, C and D, so that A's share shall be to B's as 2 to 3, B's to C's as 5 to 6, C's to D's as 8 to 9.

(18) Divide £6045 among A, B, C and D, so that A's share shall be to B's as $\frac{1}{2}$ to $\frac{2}{5}$, B's to C's as $\frac{5}{8}$ to $\frac{4}{9}$, and C's to D's as $\frac{7}{10}$ to $\frac{1}{8}$.

PART III.

APPROXIMATE CALCULATIONS.

ARITHMETIC.

PART III. APPROXIMATE CALCULATIONS.

CHAPTER I.

CONVERGING FRACTIONS.

§ 1. INSPECTING the answers to the majority of questions in Practice, Proportion, &c., where the data are not "cooked" but taken at random as they occur in the affairs of life, we find Fractions obtained with considerable labour, which are needlessly accurate. For example, on p. 65, the price of a certain quantity of goods is found to be £30. 1*s.* $5\frac{21}{320}$*d.* The most laborious part of the process consisted in finding the fraction $\frac{21}{320}$ of a penny, which, after all, cannot be paid, and is consequently disregarded. In the previous problem the result was £74. 13*s.* $2\frac{4}{7}$*d.* $\frac{4}{7}$ of a penny is more than $\frac{1}{2}$*d.* and less than $\frac{3}{4}$*d.*; it lies between these two values, but nearer to the lower; £74. 13*s.* $2\frac{1}{2}$*d.* is thus the nearest payable sum.

We may, however, easily imagine cases where the fraction could not be totally rejected, but where, nevertheless, another simpler fraction *tolerably near* the truth would sufficiently answer our purpose. The process by which we find simple expressions whose values are nearly equal to that of a more complicated one is called "Approximation."

§ 2. Examine the fraction $\frac{21}{320}$.

It is less than $\frac{1}{10}$, for $\frac{1}{10} = \frac{32}{320}$,

" $\frac{1}{12}$, for $\frac{1}{12} = \frac{26\frac{2}{3}}{320}$,

" $\frac{1}{15}$, for $\frac{1}{15} = \frac{21\frac{1}{3}}{320}$,

but it is greater than $\frac{1}{16}$, for $\frac{1}{16} = \frac{20}{320}$.

Therefore the fraction, which we will call "The Truth," lies between $\frac{1}{15}$ and $\frac{1}{16}$.

These two fractions only differ from one another by $\frac{1}{240}$; therefore neither of them differs from the truth which lies between them by so much as $\frac{1}{240}$, and each of them conveys an easier notion of the value of the fraction than does the original form. For many purposes either answer is a sufficiently close approximation; but the question may be such that $\frac{1}{240}$ is too large an error.

Assume two lines, A and B, being respectively 21 and 320 inches long. What fraction of B is A? *Ans.* $\frac{21}{320}$, which is at lowest terms, i.e. in the *simplest* form which is absolutely accurate. Dividing, nevertheless, both terms by 21, we obtain $\frac{1}{15\frac{5}{21}}$. Rejecting, for the sake of clearness, the $\frac{5}{21}$ in the denominator, we obtain $\frac{1}{15}$. This fraction must be an over-estimate, the denominator being too small. To approximate more closely to the truth, we may treat $\frac{5}{21}$ in a similar manner by dividing both terms by 5; $\frac{5}{21}=\frac{1}{4\frac{1}{5}}$, and we now obtain $\cfrac{1}{15+\cfrac{1}{4+\frac{1}{5}}}$.

Again rejecting the $\frac{1}{5}$ of the second denominator, we obtain $\frac{1}{15+\frac{1}{4}}$ $=\frac{4}{61}$. This must be an under-estimate, the second denominator being too small; and therefore, the fraction $\frac{1}{4}$ being too large, $15\frac{1}{4}$ is too great a denominator of the whole fraction, i.e. the whole fraction $\frac{1}{15+\frac{1}{4}}$ is too small. The truth then lies between $\frac{1}{15}$ and $\frac{4}{61}$, and therefore does not differ from either by so much as $\frac{1}{15}-\frac{4}{61}$ or $\frac{1}{15\times 61}$ $=\frac{1}{915}$.

The numerator of the fraction now rejected being 1, the process does not admit of a repetition; and if this fraction be taken into account, the fraction $\cfrac{1}{15+\cfrac{1}{4+\frac{1}{5}}}$ will yield $\frac{21}{320}$, the original quantity.

We have thus obtained the following approximations:

$$\left.\begin{array}{lll}\text{First value} & \frac{1}{15} & \text{an over-estimate,}\\ \text{Second} \;\; ,, & \frac{4}{61} & \text{an under-estimate,}\end{array}\right\} \begin{array}{l}\text{Limit of error,}\\ \frac{1}{15}-\frac{4}{61}=\frac{1}{915}.\end{array}$$
$$\begin{array}{lll}\text{Third} \;\;\;\; ,, & \frac{21}{320} & \text{the truth.}\end{array}$$

Take $\frac{99}{229}$. $\frac{99}{229} = \frac{99 \div 99}{229 \div 99} = \frac{1}{2 + \frac{31}{99}}$.

First approximation, $\frac{1}{2}$ an over-estimate.

$\frac{31}{99} = \frac{31 \div 31}{99 \div 31} = \frac{1}{3 + \frac{6}{31}}$.

Second approximation, $\frac{1}{2 + \frac{1}{3}} = \frac{3}{7}$ an under-estimate.

Limit of error, $\frac{1}{2} - \frac{3}{7} = \frac{1}{14}$.

$\frac{6}{31} = \frac{6 \div 6}{31 \div 6} = \frac{1}{5 + \frac{1}{6}}$.

Third approximation, $\cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{5}}} = \frac{16}{37}$ an over-estimate.

Limit of error, $\frac{16}{37} - \frac{3}{7} = \frac{1}{259}$.

Fourth approximation, $\cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{6}}}} = \frac{99}{229}$ the truth.

Examination of the several steps will shew that the successive alternate divisions are in fact nothing but the several steps of the process for finding G. C. M. by the *third method*, Part I. p. 140, the quotients being taken as the successive denominators.

Mod. op.:

$\frac{99}{229}$

2, 3, 5, 6	
99	229
6	31
—	1

$$\cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{6}}}}$$

Find a series of approximations or convergents to the fraction $\frac{2877}{7518}$.

2, 1, 1, 1, 1, 2, 2, 4	
2877	7518
1113	1764
462	651
84	189
—	21

$$\cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{4}}}}}}}}$$

			Difference between two successive estimates; hence limits of error.
First approximation,	$\frac{1}{2}$	over-estimate	
			.... $\frac{1}{6}$
Second "	$\frac{1}{3}$	under "	
			.... $\frac{1}{15}$
Third "	$\frac{2}{5}$	over "	
			.... $\frac{1}{40}$
Fourth "	$\frac{3}{8}$	under "	
			.... $\frac{1}{104}$
Fifth "	$\frac{5}{13}$	over "	
			.... $\frac{1}{442}$
Sixth "	$\frac{13}{34}$	under "	
			.... $\frac{1}{2754}$
Seventh "	$\frac{31}{81}$	over "	
Last "	$\frac{137}{358}$	the truth	

It will be observed that this last estimate does not bring back the form of the original fraction, because it was not at lowest terms. Dividing both terms of $\frac{2877}{7518}$ by 21, their G. C. M., the fraction $\frac{137}{358}$ will be obtained.

The alternation of values follows from the following considerations:

1. An increase of the numerator increases the value of the fraction.
2. An increase of the denominator decreases its value.
3. A denominator of a denominator is a numerator; but
4. The denominator of *that* is a denominator again; and so on.

Exercise I.

Find convergents and limits of error to the values of the following fractions:

(1) $\frac{181}{213}$. (2) $\frac{111}{235}$. (3) $\frac{417}{1002}$.

(4) Reduce the fraction that 1 kilometre (39,370 inches) is of an English mile to a continued fraction, and find 6 convergents.

(5) Find 4 convergents to the ratio of the diameter of a circle to its circumference 100000 : 314159.

(6) Find 5 convergents to the ratio of a lb. troy to a lb. av.

(7) Find 3 convergents to the ratio of a yard to a metre 360000 : 393708.

(8) Find 2 convergents to the ratio of a hectare ($11960\frac{1}{3}$ square yds.) to a square mile.

(9) Mont Blanc is 15,784 ft. high, and the diameter of the earth is $7912\frac{2}{5}$ miles. By what *aliquot* fraction of an inch would its height be nearly represented on a globe 18 inches in diameter?

CHAPTER II.

DECIMALS.

§ 1. We have found that the chief difficulty in fractional calculations is due to our having to manipulate fractions with different denominators. Fractions with denominators either the same or easily interconvertible present much less difficulty. The denominators 10, 100, 1000, &c., are easiest of conversion. Thus $\frac{1}{10}=\frac{10}{100}=\frac{100}{1000}$, &c.; $\frac{27}{100}=\frac{270}{1000}=\frac{2700}{10000}$, &c.

Learn by heart: *A fraction whose denominator is a power of* 10 *is called a Decimal Fraction; others are called Vulgar Fractions.*

§ 2. Decimalization of Vulgar Fractions.

Verification.

$\frac{1}{2}=\frac{1}{2}$ of $\frac{10}{10}=\frac{5}{10}$ .... $\frac{5}{10}\big|\frac{1}{2}$

$\frac{1}{4}=\frac{1}{4}$ of $\frac{10}{10}=\frac{2}{10}$ and $\frac{2}{10}$ over; $\frac{2}{10}=\frac{20}{100}$, $\frac{1}{4}$ of $\frac{20}{100}=\frac{5}{100}$, $\therefore\ \frac{1}{4}=\frac{2}{10}+\frac{5}{100}=\frac{20+5}{100}=\frac{25}{100}$ ...... $\frac{25}{100}\big|\frac{5}{20}\big|\frac{1}{4}$

$\frac{3}{4}=\frac{1}{4}$ of $3=\frac{1}{4}$ of $\frac{30}{10}=\frac{7}{10}$ and $\frac{2}{10}$ over; $\frac{2}{10}=\frac{20}{100}$, $\frac{1}{4}$ of $\frac{20}{100}=\frac{5}{100}$, $\therefore\ \frac{3}{4}=\frac{7}{10}+\frac{5}{100}=\frac{75}{100}$ .... $\frac{75}{100}\big|\frac{15}{20}\big|\frac{3}{4}$

$\frac{5}{8}=\frac{1}{8}$ of $5=\frac{1}{8}$ of $\frac{50}{10}=\frac{6}{10}$ and $\frac{2}{10}$ over; $\frac{2}{10}=\frac{20}{100}$, $\frac{1}{8}$ of $\frac{20}{100}=\frac{2}{100}$ and $\frac{4}{100}$ over; $\frac{4}{100}=\frac{40}{1000}$, $\frac{1}{8}$ of $\frac{40}{1000}=\frac{5}{1000}$, $\therefore\ \frac{5}{8}=\frac{6}{10}+\frac{2}{100}+\frac{5}{1000}=\frac{600+20+5}{1000}$ $=\frac{625}{1000}$ .... $\frac{625}{1000}\big|\frac{125}{200}\big|\frac{25}{40}\big|\frac{5}{8}$

$\frac{137}{512}=\frac{1}{512}$ of $137=\frac{1}{512}$ of $\frac{1370}{10}=\frac{2}{10}$ and $\frac{346}{10}$ over; $\frac{346}{10}=\frac{3460}{100}$, $\frac{1}{512}$ of $\frac{3460}{100}=\frac{6}{100}$ and $\frac{388}{100}$ over; $\frac{388}{100}=\frac{3880}{1000}$, &c.

Mod. op.:

$$
\begin{array}{l}
512)1370\ (\frac{2}{10}+\frac{6}{100}+\frac{7}{1000}+\frac{5}{10000}+\frac{7}{100000}+\frac{8}{1000000}\\
\quad 3460 \qquad +\frac{1}{10000000}+\frac{2}{100000000}+\frac{5}{1000000000}\\
\quad\ 3880\\
\quad\ \ 2960\\
\quad\ \ \ 4000\\
\quad\ \ \ \ 4160\\
\quad\ \ \ \ \ \ 640\\
\quad\ \ \ \ \ \ 1280\\
\quad\ \ \ \ \ \ \ 2560\\
\quad\ \ \ \ \ \ \ \ \cdots\cdot
\end{array}
$$

Ans. $\frac{2}{10}+\frac{6}{100}+\frac{7}{1000}+\frac{5}{10000}+\frac{7}{100000}+\frac{8}{1000000}+\frac{1}{10000000}+\frac{2}{100000000}+\frac{5}{1000000000}$

$=\frac{200000000+60000000+7000000+500000+70000+8000+100+20+5}{1000000000}$

$=\frac{267578125}{1000000000}$.

Verification: $\frac{267578125}{1000000000}\big|\frac{53515625}{200000000}\big|\frac{10703125}{40000000}\big|\frac{2140625}{8000000}\big|\frac{428125}{1600000}\big|\frac{85625}{320000}\big|\frac{17125}{64000}\big|\frac{3425}{12800}\big|\frac{685}{2560}\big|\frac{137}{512}$.

Exercise II.

Reduce to decimal fractions and verify the results:

(1) $\frac{1}{5}$.	(4) $\frac{4}{5}$.	(7) $\frac{5}{8}$.	(10) $\frac{13}{20}$.	(13) $\frac{17}{64}$.
(2) $\frac{2}{5}$.	(5) $\frac{1}{8}$.	(8) $\frac{7}{8}$.	(11) $\frac{2}{25}$.	(14) $\frac{119}{128}$.
(3) $\frac{3}{5}$.	(6) $\frac{3}{8}$.	(9) $\frac{5}{16}$.	(12) $\frac{17}{40}$.	(15) $\frac{39}{125}$.

§ 3. In reducing $\frac{137}{512}$ to a decimal fraction, the denominator of any figure of the result may be known at once by knowing the place of that figure; thus the first figure has for denominator, 1 followed by one nought; the second, 1 followed by two noughts; the seventh, 1 followed by seven noughts, &c.; hence if the quotient be written 267578125, and if we know that the 2 is 2 tenths, the denominators of all the other figures will be known and may be omitted. To distinguish decimals from integers, a point is placed after the units' figure; thus 4·7 means 4 wholes and $\frac{7}{10}=4\frac{7}{10}$; similarly 123·456 means 123 units $+\frac{4}{10}+\frac{5}{100}+\frac{6}{1000}=123\frac{456}{1000}=\frac{123456}{1000}$. The absence of a numerator to any denominator must be indicated by a cipher; thus:

a. $\cdot 42=\frac{4}{10}+\frac{2}{100}=\frac{42}{100}$.
b. $\cdot 402=\frac{4}{10}+\frac{0}{100}+\frac{2}{1000}=\frac{402}{1000}$.
c. $\cdot 042=\frac{0}{10}+\frac{4}{100}+\frac{2}{1000}=\frac{42}{1000}$.
d. $\cdot 420=\frac{4}{10}+\frac{2}{100}+\frac{0}{1000}=\frac{420}{1000}=\frac{42}{100}$.

§ 4. Line *a* is of the same value as line *d*; line *c* is one-tenth of line *a*; from which it appears that a 0 added to either end of a decimal fraction produces an effect very different from that which it produces on integers.

Integers.	Decimals.
0743 = 743	·7430 = ·743
7430 = 10 × 743	·0743 = $\frac{1}{10}$ of ·743

In words: In integers, a 0 on the left is of no effect; in decimals, a 0 on the right is of no effect. In integers, a 0 on the right *multiplies* by 10; in decimals, a 0 on the left *divides* by 10.

Read off: ·007. *Ans.* $\frac{7}{1000}$.

" ·04051. " $\frac{4}{100}+\frac{5}{10000}+\frac{1}{100000}=\frac{4051}{100000}$.

" 16·25. " 16 wholes $+\frac{2}{10}+\frac{5}{100}=16\frac{25}{100}=\frac{1625}{100}$.

Exercise III.

I. Give the different readings of:

(1) ·1.	(6) 1·01.	(11) 3·15.
(2) ·01.	(7) ·11.	(12) 31·5.
(3) ·001.	(8) ·31.	(13) 128·053.
(4) ·000001.	(9) ·315.	(14) 12·8053.
(5) 1·1.	(10) ·0315.	(15) 1280·53.

II. Reduce to vulgar fractions at lowest terms:

(1) ·785.	(6) 6·03125.	(11) ·00055.
(2) ·1875.	(7) 603·125.	(12) ·505.
(3) ·73125.	(8) ·55.	(13) 5·05.
(4) ·603125.	(9) ·371.	(14) 8·888.
(5) ·128.	(10) ·00016.	(15) ·728.

§ 5. Addition.

Simplify ·68 + 43·159 + ·07 + 9·00124 + 453·69 + 8·9871 + 412.

Mod. op.:

```
   ·68
 43·159
   ·07
  9·00124
453·69
  8·9871
412
─────────
927·58734
```

Arrange the addenda in column, placing units under units, tenths under tenths, &c. As the decimal point marks the units' place, the figures will fall into their proper places if the decimal points are under one another. Since the figures are arranged in the decimal scale, the same rules for carrying hold as in integers.

Add by vulgar fractions and by decimals, $2\frac{3}{5}$, $4\frac{5}{8}$, $7\frac{11}{20}$, $49\frac{13}{25}$.

By vulgar fractions:

L.C.M. 200

2	40	120
4	25	125
7	10	110
49	8	104
62	200) 459(2	
2	59	

Ans. $64\frac{59}{200}$.

By decimals:

$2\frac{3}{5} = 2{\cdot}6$
$4\frac{5}{8} = 4{\cdot}625$
$7\frac{11}{20} = 7{\cdot}55$
$49\frac{13}{25} = 49{\cdot}52$

Ans. $64{\cdot}295$

$\frac{295}{1000} \mid \frac{59}{200}$.

Ans. By either method, $64\frac{59}{200}$.

Exercise IV.

Simplify by vulgar fractions and by decimals, shewing that the results agree:

(1) $7\frac{2}{5} + 4\frac{5}{8} + 9\frac{13}{20} + 11\frac{29}{32}$.
(2) $84\frac{13}{20} + 19\frac{11}{25} + 417\frac{19}{32} + 5043\frac{49}{64} + \frac{41}{50}$.
(3) $4\frac{27}{64} + 13\frac{17}{20} + 42\frac{37}{50} + 418\frac{19}{25} + 2\frac{13}{16} + 1\frac{1}{2}$.
(4) $5\frac{7}{8} + 13\frac{4}{5} + 19\frac{7}{16} + 7\frac{3}{20} + 18\frac{17}{40}$.
(5) $37\frac{5}{16} + 9\frac{4}{5} + \frac{2}{8}$ of $1\frac{4}{5} + \frac{7}{8}$ of $2\frac{2}{7} + \frac{3}{4}$ of $\frac{5}{8}$ of $\frac{7}{10}$.
(6) $9\frac{11}{32} + \frac{47}{100} + 11\frac{19}{128} +.3\frac{3}{5}$ of $4\frac{1}{9} + (\frac{2}{3} - \frac{1}{6})$.

§ 6. Subtraction.

From 17·08 take 9·643.

Mod. op.:

```
 17·08
  9·643
 ------
  7·437
```

Subtract as in integers; where there is no figure in the minuend over one in the subtrahend, place or imagine a 0. (Cf. § 4.)

Exercise V.

Simplify by vulgar fractions and by decimals:

(1) $7\frac{2}{5} - 4\frac{5}{8}$.
(2) $84\frac{13}{20} - 17\frac{31}{32}$.
(3) $100\frac{2}{25} - 83\frac{17}{80}$.
(4) $100 - 17\frac{113}{625}$.
(5) $82\frac{1}{5} - 37\frac{11}{16}$.
(6) $5\frac{1}{2} - \frac{3}{4}$ of $1\frac{13}{24}$.
(7) $8\frac{1}{5} - 1\frac{1}{2}$ of $\frac{3}{16}$.
(8) $\frac{14}{25} - \frac{11}{64}$.

§ 7. MULTIPLICATION.

CASE I. By a power of 10.

$$457{\cdot}6843 \times 10 = \tfrac{4576843}{10000} \times 10 = \tfrac{4576843}{1000} = 4576{\cdot}843.$$

Comparing the product with the multiplicand, we see that the figures are the same, the only difference being that the decimal point is one place further to the right, so that each figure is one place higher in the decimal scale, i.e. is ten times as valuable.

To multiply by 10, we have therefore only to shift the decimal point one place to the right; by 100, two places; by 1000, three places, and so on.

Rule: To multiply a decimal by any power of 10, shift the decimal point as many places to the *right* as there are ciphers in the multiplier, adding ciphers if necessary. Thus:

$$7{\cdot}63 \times 100000 = 763000.$$

EXERCISE VI.

By vulgar fractions and by decimals:

(1) $4\frac{7}{25} \times 10$, 100, 1000, 10000.

(2) $56\frac{19}{32} \times 10$, 100, 1000, 10000.

(3) $8\frac{11}{16} \times 10$, 100, 1000.

(4) $\frac{19}{64} \times 10$, 100, 1000, 10000000.

CASE II. By any integer.

a. $457{\cdot}6843 \times 9 = \tfrac{4576843}{10000} \times 9 = \tfrac{41191587}{10000} = 4119{\cdot}1587.$

b. $74{\cdot}9375 \times 8 = \tfrac{749375}{10000} \times 8 = \tfrac{5995000}{10000} = 599{\cdot}5000 = 599{\cdot}5.$

c. $\cdot00731 \times 7000 = \cdot00731 \times 1000 \times 7 = 7{\cdot}31 \times 7 = 51{\cdot}17.$

In each case, the multiplicand being ten-thousandths, the product must also be ten-thousandths; in other words, the multiplicand having four decimal places, the product must have the same number of places. When the place of the decimal point is determined, ciphers to the right of the product may be rejected as superfluous.

Rule: To multiply a decimal by an integer, proceed as in common multiplication, and from the product mark off (counting from the right) as many decimal places as there are in the multiplicand.

EXERCISE VII.

By vulgar fractions and by decimals:

(1) $43\frac{27}{50} \times 7$; $435\frac{2}{5} \times 7$; $\frac{2177}{5000} \times 7$; $4\frac{177}{500} \times 7$.

(2) $47\frac{5}{8} \times 3$, 7, 37, 4, 6, 8, 46, 468.

(3) $\frac{315}{512} \times 5043$, 64.

By decimals only:

(4) $9000 \times 167{\cdot}432$, $\cdot 00719$, $\cdot 000001$.

(5) $\cdot 0678 \times 512000$; $\cdot 03625 \times 102400$.

CASE III. By a decimal.

a. $\cdot 43 \times 3{\cdot}784$.

$$\cdot 43 \times 3{\cdot}784 = \frac{43}{100} \times \frac{3784}{1000} = \frac{43 \times 3784}{100 \times 1000} = \frac{162712}{100000} = 1{\cdot}62712.$$

b. $\cdot 03 \times \cdot 0005$.

$$\cdot 03 \times \cdot 0005 = \frac{3}{100} \times \frac{5}{10000} = \frac{15}{1000000} = \cdot 000015.$$

c. $3{\cdot}175 \times 25{\cdot}6$.

$$3{\cdot}175 \times 25{\cdot}6 = \frac{3175}{1000} \times \frac{256}{10} = \frac{3175 \times 256}{1000 \times 10} = \frac{812800}{10000} = 81{\cdot}2800 = 81{\cdot}28.$$

From these three instances the following rule is obvious:

To multiply a decimal by a decimal, proceed as in common multiplication, and from the product mark off (counting from the right) as many decimal places as there are in the multiplier and multiplicand together. Any ciphers on the right of the product may be struck out *after* the position of the decimal point is determined. This rule evidently includes those given for Cases I. and II.

EXERCISE VIII.

(1) $16{\cdot}42 \times 4{\cdot}17$.

(2) $1{\cdot}642 \times 41{\cdot}7$.

(3) $\cdot 1642 \times 417$.

(4) $164{\cdot}2 \times \cdot 0417$.

(5) $\cdot 3 \times \cdot 4$; $\cdot 03 \times \cdot 004$; $\cdot 03 \times 1$; $\cdot 03 \times \cdot 1$; $\cdot 003 \times \cdot 001$; $\cdot 005 \times \cdot 04$; $\cdot 004 \times \cdot 05$.

(6) $1{\cdot}1 \times \cdot 011$; $1{\cdot}01 \times \cdot 0101$; $\cdot 04 \times \cdot 04 \times \cdot 05$; $\cdot 1 \times \cdot 1 \times \cdot 01$; $\cdot 7 \times \cdot 4 \times \cdot 3 \times 1000$.

(7) $72{\cdot}159 \times 3{\cdot}27$; $7{\cdot}2159 \times \cdot 327$; $\cdot 72159 \times \cdot 0327$.

(8) $16{\cdot}875 \times 5{\cdot}12$; $1{\cdot}6875 \times 51{\cdot}2$; $\cdot 16875 \times \cdot 0512$.

By vulgar fractions and by decimals :

(9) $4\frac{5}{8} \times 1\frac{1}{2}$; $7\frac{4}{5} \times \frac{3}{16}$; $\frac{13}{64} \times \frac{1}{25}$.

(10) $4\frac{7}{25} \times \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{25}$.

(11) $4\frac{7}{25} \times \cdot02, \cdot03, \cdot04, \cdot05, \cdot032, \cdot302, 3\cdot2, \cdot00032$.

§ 8. Division.

Case I. By a power of 10.

$457\cdot6843 \div 10$.

$$\frac{4576843}{10000} \div 10 = \frac{4576843}{100000} = 45\cdot76843.$$

Comparing the quotient with the dividend, we see that the figures are the same, the only difference being that the decimal point is one place further to the left, so that each figure is one place lower in the decimal scale, i. e. it has one-tenth of the value.

To divide by 10, we have therefore only to shift the point one place to the left; by 100, two places; by 1000, three places, and so on.

Rule : To divide a decimal by any power of 10, shift the decimal point as many places to the *left* as there are ciphers in the divisor, prefixing ciphers if necessary. Thus,

$$7\cdot63 \div 100000 = \cdot0000763.$$

Exercise IX.

By vulgar fractions and by decimals :

(1) $32\frac{5}{16} \div 10, 100, 1000, 10000$.

(2) $4\frac{7}{25} \div 10, 100, 1000$.

(3) $56\frac{19}{32} \div 10, 100$.

Case II. By any integer.

a. $4379 \div 5$.

$$\begin{array}{r} 5)\underline{4379} \\ 875\cdot8 \end{array}$$

Wording: 5 in 43, 8′, carry 3; in 37, 7′, carry 2; in 29, 5′, carry 4 (units) = 40 (tenths); in 40, 8′ (tenths).

Verification :

$$\begin{array}{r} 875\cdot8 \\ \underline{5} \\ 4379\cdot0 \end{array}$$

b. $3101 \div 8.$

$$\begin{array}{r} 8\underline{)3101} \\ 387{\cdot}625 \end{array}$$

Wording: 8 in 31, **3**′, carry 7; in 70, **8**′, carry 6; in 61, **7**′, carry 5; in 50, **6**′, carry 2; in 20, **2**′, carry 4; in 40, **5**′.

Verification:

$$\begin{array}{r} 387{\cdot}625 \\ \underline{8} \\ 3101{\cdot}000 \end{array}$$

c. $123{\cdot}75 \div 4.$

$$\begin{array}{r} 4\underline{)123{\cdot}75} \\ 30{\cdot}9375 \end{array}$$

Wording: 4 in 12, **3**′; in 3, **0**′, carry 3; in 37 (tenths), **9**′, carry 1; in 15, **3**′, carry 3; in 30, **7**′, carry 2; in 20, **5**′.

Verification:

$$\begin{array}{r} 30{\cdot}9375 \\ \underline{4} \\ 123{\cdot}7500 \end{array}$$

d. $4{\cdot}19 \div 800 = (4{\cdot}19 \div 100) \div 8 = {\cdot}0419 \div 8.$

$$\begin{array}{r} 8\underline{)\,{\cdot}0419} \\ {\cdot}0052375 \end{array}$$

Wording: 8 in 0 (tenths), **0**′; in 4, **0**′; in 41, **5**′, carry 1; in 19, **2**′, carry 3; in 30, **3**′, carry 6; in 60, **7**′, carry 4; in 40, **5**′.

Verification: $\quad {\cdot}0052375 \times 800 = 4{\cdot}19000.$

e. ${\cdot}0143 \div 32.$

$$\begin{array}{r} 8\underline{)\,{\cdot}0143} \\ 4\underline{)\,{\cdot}0017875} \\ {\cdot}000446875 \end{array}$$

Verification: $\quad {\cdot}000446875 \times 32 = {\cdot}0143.$

f. $5{\cdot}643 \div 76.$

$$\begin{array}{l} 76)5{\cdot}643\,({\cdot}07425 \\ \quad\;\; 323 \\ \quad\;\; 190 \\ \quad\;\;\; 380 \\ \quad\;\;\; \ldots \end{array}$$

From these six instances the following rule is obvious: To divide a decimal by an integer proceed as in common division, placing the decimal point in the quotient as soon as the figure in the first place of decimals is "brought down," and adding ciphers to the successive remainders as required.

N.B. The examples given in this section must be "carried out" until there is no remainder.

Exercise X.

(Verify all these by multiplication.)

(1) $749{\cdot}682 \div 2, 4, 5, 8$.
(2) $32594{\cdot}73 \div 32, 128, 25, 1024$.
(3) $358677{\cdot}9 \div 99288$.
(4) $\cdot 1 \div 64, 512, 50, 800$.
(5) $\cdot 13 \div 52, 10400$.
(6) $\cdot 5 \div 2, 4, 5, 8$.
(7) $\cdot 01 \div 2, 4, 5, 8$.
(8) $\cdot 0073 \div 16, 1600, 160000$.

By vulgar fractions and by decimals:

(9) $28\frac{11}{40} \div 2, 3, 4, 5, 8, 58$.
(10) $13 \div 80$; $429 \div 16000$; $8193 \div 163840000$.

Case III. By a decimal.

By multiplying the dividend, the quotient is multiplied.
By multiplying the divisor, the quotient is divided.
By multiplying both dividend and divisor by the same number, the quotient remains unaltered. (But the remainder, if any, is still multiplied, being unaffected by the multiplication of the divisor.)
Hence to divide by a decimal, first of all multiply both divisor and dividend by such a power of 10 as will expel the decimal point from the divisor. This will reduce the problem to one under Case II.

$12{\cdot}307722 \div {\cdot}294 = 12307{\cdot}722 \div 294$.

```
294)12307·722(41·863
     547
     2537
      1852
        882
        ...
```

$\cdot 005 \div 3{\cdot}2 = \cdot 05 \div 32$.

```
8)·05
4)·00625
  ·0015625
```

EXERCISE XI.

By vulgar fractions and by decimals:

(1) $\cdot 0073 \div \frac{4}{25}$.
(2) $\cdot 1 \div \frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{4}{5}$.
(3) $93\frac{3}{5} \div \cdot 2, \cdot 3, \cdot 4, \cdot 5, \frac{4}{5}$.
(4) $417\cdot 143 \div 12\frac{4}{5}, \frac{16}{125}, \frac{4}{3125}$.
(5) $\frac{4}{3125} \div 8\frac{24}{125}$.
(6) $1708\cdot 4592 \div \cdot 00024$.
(7) $6\frac{9}{16} \div 1\frac{1}{2}$.
(8) $6\frac{9}{16} \div 4\frac{3}{8}$.
(9) $28\frac{11}{40} \div \cdot 58, \cdot 058, 4\cdot 875, \cdot 4875, 48\cdot 75$.
(10) $\cdot 1 \div \cdot 1$; $\cdot 1 \div 1$; $1 \div \cdot 1$; $\cdot 01 \div \cdot 0004$; $\cdot 0004 \div \cdot 01$.

§ 9. Find the value of ·06875 of a cwt.

1st method.

$$\cdot 06875 = \frac{6875}{100000} = \frac{11}{160}\text{ cwt.} = \frac{11 \times \overset{7}{\cancel{112}}}{\underset{10}{\cancel{160}}}\text{ lbs.} = \frac{77}{10} = 7\frac{7}{10}\text{ lbs.}$$

Ans. $7\frac{7}{10}$ lbs.

2nd method.

$$\begin{array}{r} \cdot 06875 \text{ cwt.} \\ 4 \\ \hline \cdot 275\cancel{0}\cancel{0} \text{ qrs.} \\ 4 \\ \hline 1\cdot 1\cancel{0}\cancel{0} \\ 7 \\ \hline 7\cdot 7 \text{ lbs.} \end{array}$$

Ans. $7\frac{7}{10}$ lbs.

Find the value of £·75003125.

1st method. $£\cdot 75003125 = £\frac{75003125}{100000000} = £\frac{24001}{32000} = 15s.\ 0\frac{3}{400}d.$

2nd method.

$$\begin{array}{r} £\cdot 75003125 \\ 20 \\ \hline 15\cdot 000625\cancel{0} \text{ shillings} \\ 12 \\ \hline \cdot 0075\cancel{0}\cancel{0} = \frac{75}{10000} = \frac{3}{400}d. \end{array}$$

Ans. $15s.\ 0\frac{3}{400}d.$

EXERCISE XII.

(1) Find the value of £·721875.
(2) " ·6 of 1s.
(3) " ·045 of 1 cwt.

(4) Find the value of ·8 of a year.
(5) „ ·2345 of an hour.
(6) „ ·0109375 tons.
(7) „ ·06412 miles.

§ 10. Reduce $1\frac{3}{4}$ lbs. to the decimal of 1 cwt.

1st method. $1\frac{3}{4}$ lbs. $= \frac{7}{4}$ lbs. $= \dfrac{\overset{1}{\cancel{7}}}{4 \times \underset{16}{\cancel{112}}}$ cwt. $= \frac{1}{64}$ cwt.

$$\begin{array}{l} 8\underline{)1\cdot\quad} \\ \;8\underline{)\cdot125} \\ \quad\cdot015625 \text{ cwt.} \end{array}$$

Ans. ·015625 cwt.

2nd method.

$$\begin{array}{l} 4\underline{)7\quad} \\ 4\underline{)1\cdot75} \text{ lbs.} \\ \;7\underline{)\cdot4375} \\ \;\;4\underline{)\cdot0625} \text{ qrs.} \\ \quad\cdot015625 \text{ cwt.} \end{array}$$

Ans. ·015625 cwt.

Exercise XIII.

(1) Reduce 146 days to the decimal of 1 year.
(2) „ 77 lbs. „ 1 ton.
(3) „ 4 dwts., 15 grs. „ 1 oz. troy.
(4) „ 7 cwt., 3 qrs., $17\frac{1}{2}$ lbs. „ 1 cwt.
(5) „ 11*s.* $5\frac{1}{4}$*d.* „ £1.
(6) „ £4. 13*s.* $9\frac{3}{4}$*d.* „ £100.

§ 11. Simplification of Complex Decimals.

Simplify $\dfrac{4\cdot375}{4\cdot5} \times \dfrac{41}{5\cdot125} \times \dfrac{7\cdot875}{\cdot0625}$.

The product of the numerators will have 6 places, that of the denominators 8. If we expel all the points, treating the several numbers as integers, we shall have to multiply the result by 100. This gives:

$$\frac{\overset{\overset{\overset{\overset{7}{\cancel{35}}}{\cancel{175}}}{\cancel{875}}}{\cancel{4375}} \times \cancel{41} \times \overset{\overset{\overset{\overset{7}{\cancel{63}}}{\cancel{315}}}{\cancel{1575}}}{\cancel{7875}} \times 100}{\underset{\cancel{9}}{\cancel{45}} \times \underset{\underset{\underset{\cancel{41}}{\cancel{205}}}{\cancel{1025}}}{\cancel{5125}} \times \underset{\underset{\underset{5.}{\cancel{25}}}{\cancel{125}}}{\cancel{625}}} = \frac{4900}{5} = 980.$$

Simplify $\frac{4\cdot75}{5\cdot625}$. $\frac{4\cdot75}{5\cdot625}=\frac{4\cdot75\times8}{5\cdot625\times8}=\frac{38\cdot\cancel{00}}{45\cdot\cancel{000}}$.

A terminal 5 in a decimal will disappear by multiplication by 2.
„ 25 or 75 „ „ 4.
„ 125, 375, 625 or 875 „ „ 8.
(Cf. Part I. Ch. IX. § 19.)

This, applied to the former simplification, gives:

$$\frac{4\cdot375\times8}{4\cdot5\times8}\times\frac{41\times8}{5\cdot125\times8}\times\frac{7\cdot875\times8}{\cdot0625\times8}=\frac{35\times\cancel{41}\times8\times63}{36\times\cancel{41}\times\cdot5}=\frac{35\times\overset{2}{\cancel{8}}\times\overset{7}{\cancel{63}}\times2}{\underset{\cancel{4}}{\cancel{36}}\times\cancel{\cdot5}\times\cancel{2}}=980.$$

Reduce to a vulgar fraction at lowest terms ·267578125.

$\frac{267578125\times8}{1000000000\times8}=\frac{2140625\cancel{000}\times8}{8000000\cancel{000}\times8}=\frac{17125\cancel{000}\times8}{64000\cancel{000}\times8}=\frac{137\cancel{000}}{512\cancel{000}}$. (Cf. Part III. Ch. II. § 2.) *Ans.* $\frac{137}{512}$.

§ 12. G.C.M. AND L.C.M. OF DECIMALS.

Find G.C.M. and L.C.M. of 5·625, 2·88, 3·6, 2·8125.

Equalize the decimal places: 5·6250, 2·8800, 3·6000, 2·8125.

Find G.C.M. and L.C.M. of these numbers, disregarding decimal points, and mark off the 4 places from each result. (Cf. Part II. Ch. IV. § 4.) *Ans.* G.C.M. ·0225; L.C.M. 360.

CHAPTER III.

THE METRIC SYSTEM.

Previous to the French Revolution of 1789, France was divided into provinces, many of which had their own peculiar weights, measures, and even coinage. The Revolution introduced (March, 1795) a uniform system, which was, however, adopted slowly and

with difficulty. It only acquired universally legal force in 1840. It is now in use in its entirety in France and Belgium, Italy and Germany.

In this system, the distance from the North Pole to the Equator (the Quadrant) was divided into 10,000,000 equal parts, of which each was called a METRE (39·37079 inches). This length forms the unit of the whole system, and hence the name, "Metric System." Multiples and parts of this unit run decimally, i. e. by powers of 10; hence the system is a decimal system. These two features of the system, its choice of unit and its decimal character, are quite independent of one another, and each might exist without the other.

The Greek prefixes, *deca*, *hecto*, *kilo* and *myria*, are used to indicate decimal multiplication of the metre; and the Latin prefixes, *deci*, *centi*, *milli*, to indicate decimal subdivisions of the metre. Thus: LENGTH.

Myriametre	=	393707·9	Inches.
Kilometre	=	39370·79	
Hectometre	=	3937·079	
Decametre	=	393·7079	
METRE	=	39·37079	
Decimetre	=	3·937079	
Centimetre	=	·3937079	
Millimetre	=	·03937079	

SURFACE. The unit of Surface is a decametre square, and is called the ARE (= ·02471143 acres).

10000 square metres	=	1 hectare	= 2·471143	acres.
100 ,, metres	=	1 ARE	= ·02471143	,,
1 ,, metre	=	1 centiare	= ·0002471143	,,

SOLIDITY. The unit of Solidity is a cubic metre, and is called a STERE (= 35·32 cubic feet).

CAPACITY. The unit of Capacity is a cubic decimetre, and is called a LITRE (= ·22009687 imperial gallons).

Kilolitre	=	220·09687	Gallons.
Hectolitre	=	22·009687	
Decalitre	=	2·2009687	
LITRE	=	·22009687	
Decilitre	=	·022009687	
Centilitre	=	·0022009687	
Millilitre	=	·00022009687	

WEIGHT. The unit of weight is a cubic centimetre of distilled water at its maximum density (very nearly 40° F.), and is called 1 GRAMME = ·00220606 lbs. av., 15·442 grains.

Ton or Millier	=	2204·62 lbs. av. = ·9842 tons.	
Quintal	=	220·462	lbs. av.
Myriagramme	=	22·0462	lbs. av.
Kilogramme	=	2·20462	lbs. av.
Hectogramme	=	1543·23487	grains.
Decagramme	=	154·32349	grains.
GRAMME	=	15·43235	grains.
Decigramme	=	1·54323	grains.
Centigramme	=	·15432	grains.
Milligramme	=	·01543	grains.

MONEY. 5 grammes of silver of a certain fixed fineness are coined into 1 FRANC = 10 decimes = 100 centimes = $9\frac{1}{2}$ pence* nearly. (In familiar language, the decime is called *deux sous*).

Hence 7·485 metres = 7 metres, 4 decimetres, 8 centimetres and 5 millimetres; or 7 metres, 485 millimetres, &c.

14·89 litres = 14 litres, 8 decilitres, 9 centilitres; or 14 litres, 89 centilitres, &c., and so on.

Find the cost of 10, 100, 1000, 10,000 metres, at fr. 2·35 each.

1 metre	costs	fr. 2·35
10	,,	23·5
100	,,	235
1000	,,	2350
10000	,,	23500

Find the cost of 1 kilogramme at fr. 23·5 per quintal.

1 quintal costs fr. 23·5
1 kilogramme ,, ·235 = $23\frac{1}{2}$ centimes.

Find the cost of 385·75 metres at fr. 4·65 per metre.

$$\begin{array}{r} 38575 \\ \underline{465} \\ 192875 \\ 231450 \\ \underline{154300} \\ 1793{\cdot}7375 \end{array}$$

Ans. fr. 1793·73.

N.B. Fractions of a centime are disregarded.

* £1 sterling at par = fr. 25·2215.

Find the cost of 273·42 litres at fr. 123·5 per hectolitre.

1 hectolitre costs fr. 123·5, ∴ 1 litre costs fr. 1·235.

```
  27342
   1235
 ------
 136710
 82026
54684
27342
---------
337·67370
```

Ans. fr. 337·67.

Find the cost in English money of 17 litres, if 1 hectolitre costs fr. 454·5, and £1 = fr. 25·25.

```
      £ x | 17 litres.
litres 100 | 454·5 fr.
 fr. 25·25 | £1.
```

```
·06                 4545 × 17
  20          2525) 7726·5 (3·06
----                151 50
1·2                 • • • • •
 12
----
2 4/10 = 2/5
```

Ans. £3·06 = £3. 1s. $2\frac{2}{5}$d.

Find the cost per litre, if 468·59 litres cost fr. 1274.

```
46859)127400(2·718
      336820
        88070
       412110
        372380
```

Ans. fr. 2·72 nearly.

Exercise XIV. (*Miscellaneous.*)

(1) Find the sum of 4·173, ·0089, ·2375, ·1, ·01, 246.

(2) What number exceeds ·999 by ·001?

(3) From what vulgar fraction must ·625 be subtracted to leave ·295?

(4) By vulgar fractions and by decimals, find $4\frac{7}{8}$ + ·01375.

(5) ·0876 exceeds a certain quantity by ·00876. Find it.

(6) There are two numbers; the greater is 3·142857; their difference is ·001267. Find the less.

(7) If the year is reckoned at $365\frac{1}{4}$ days instead of 365·242264 days, what will be the amount of error in 19 centuries?

(8) Express ·4984 of a day in hours, minutes and seconds.

(9) What fraction contains ·125 ·486 times?

(10) Of what number is ·4 the 25th part?

(11) How many times can ·0085 be subtracted from ·18, and what will be over?

(12) How many times can ·029 be taken out of ·3786, and what will be over?

(13) How much must be subtracted from ·710267875 to leave the largest multiple of ·000000046275 it contains?

(14) Simplify $(7{\cdot}13 + 3{\cdot}875) + (7{\cdot}13 - 3{\cdot}875) + (7{\cdot}13 \times 3{\cdot}875) + (7{\cdot}13 \div 3{\cdot}875)$.

(15) Find the reciprocal of ·64.

(16) $72{\cdot}315 \times 1000$, $\cdot 001$, $\frac{1}{100}$, $\frac{1}{\cdot 001}$.

(17) $72{\cdot}315 \div 1000$, $\cdot 001$, $\frac{1}{100}$, $\frac{1}{\cdot 001}$.

(18) Find the cost of 12·5 kilolitres at 3·75 francs per litre.

(19) Find the cost per metre if 437·75 metres cost 1805·71875 francs.

(20) Find the value of £·021875 + ·375*s.* + 4·75*d.*

(21) Find the value of £·08 + ·08*s.* + ·09*d.*, and express the result as a decimal of £1.

(22) Add together £1·3625, ·75 of 13*s.* 4*d.*, and $\frac{2}{25}$ of £20.

(23) Reduce ·06 of ·42 of a guinea to the decimal of £1.

(24) Divide ·010875 by ·00625, and verify by vulgar fractions.

(25) Reduce to a vulgar fraction $\cdot 7 + \frac{3}{11}$ of $\cdot 825 + 4{\cdot}13$.

(26) Simplify:

a. $\dfrac{1{\cdot}4 \times \cdot 035}{\cdot 00014}$.

b. $\dfrac{1{\cdot}4 + \cdot 035}{\cdot 00014}$.

c. $\dfrac{1{\cdot}4 - \cdot 035}{\cdot 00014}$.

d. $\dfrac{\cdot 1 \times \cdot 01 \times \cdot 001 \times \cdot 001}{\cdot 001 \times \cdot 0001}$.

e. $\dfrac{\cdot 13 \times \cdot 14 \times \cdot 01 + \cdot 12 \times \cdot 13 \times \cdot 01 - \cdot 12 \times \cdot 14 \times \cdot 02}{\cdot 01 \times \cdot 2 \times \cdot 01}$,

f. $\dfrac{1{\cdot}85}{29{\cdot}6} + \dfrac{\cdot 51}{\cdot 425}$ of $\dfrac{6{\cdot}875}{22} + \cdot 0625$.

(27) Reduce to metres:

17·35 myriametres,	17·35 decimetres,
17·35 kilometres,	17·35 centimetres,
17·35 hectometres,	17·35 millimetres.

(28) How many square decimetres in a square metre, and how many cubic centimetres in a cubic metre?

(29) Express 500 cubic metres as cubic yards, taking 1 metre = 39·37 inches.

(30) Find G.C.M. and L.C.M. of 64·09 and 7·395.

(31) A sum of money is divided among three persons; the first receives ·375 of the whole, the second ·6, and the third £2·125. Find it.

(32) A does ·375 of a piece of work in 2·25 days, and B does the remainder of it in 3·75 days. How many such pieces of work would A and B together do in 12 days?

(33) Find the value of the following: ·175 tons + ·195 cwt. + ·145 qr. + ·15 lbs. + ·2 oz.

(34) If a gramme is 15·442 grains, and a metre 39·37 inches, how many grammes are there in 1000 grains, and how many metres in a mile?

(35) Simplify $\dfrac{\cdot 002 \times 1\cdot 75 \div \cdot 00007}{1\frac{3}{8} \div \frac{1}{8}}$.

(36) Find the sum, difference, product and quotient (the greater being divided by the less) of ·016 and ·02235.

(37) Which is more, and by how much, £·4999 or £·5?

(38) Find the cost of 1 millier at fr. 1·75 per kilog.

(39) Find the cost in English money per lb. av., if 950 grammes* cost fr. 21·23, and fr. 1 = $9\frac{1}{2}d$.

(40) Express in English units: $7\frac{1}{2}$ quintals, 580 kiloms., 85·6 ares, $1437\frac{1}{2}$ francs,† 570 litres.

(41) Reckoning £1 = fr. 25·2215, find the value in French money of £450.

* 1 gramme = 15·43 grains. † At $9\frac{1}{2}d$. per fr.

CHAPTER IV.

RECURRING DECIMALS.

§ 1. Reduce $\frac{1}{3}$ to a decimal.

$$3\overline{)1\cdot}$$
$$\cdot 333, \&c.$$

We see that $\frac{1}{3}$ cannot be accurately expressed as a decimal, and the question arises, Are there many such fractions? If by trial we classify all the fractions from $\frac{1}{2}$ to $\frac{1}{16}$, we shall find that

$\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{16}$, are reducible;

$\frac{1}{3}, \frac{1}{6}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}$, are not reducible;

and if the trial is carried further, we shall find that the preponderance of non-reducible fractions continually increases. Either, then, decimal fractions are comparatively useless, being only applicable to a very few fractions, or methods must be found for manipulating non-reducible fractions.

In the reduction of $\frac{1}{3}$, we find that the quotient 3 continually recurs, and this is expressed thus, $\cdot\dot{3}$; similarly, ·575757, &c., is written $\cdot\dot{5}\dot{7}$, and ·41374137, &c., is written $\cdot\dot{4}13\dot{7}$. Sometimes only a part of the quotient will recur, as in ·6741094109, &c., which is written $\cdot 67\dot{4}10\dot{9}$, the dots being placed over the first and last figures of the recurring "period." Decimal fractions where ALL the figures recur are called PURE CIRCULATORS; those where some of the figures do not recur are called MIXED CIRCULATORS.

§ 2. In dividing 1 by 3, we obtained for quotient $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$ ad infinitum, with the successive remainders, $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$, &c., also ad infinitum. Wherever we stop, we disregard this remainder, and therefore obtain an inaccurate result. This inaccuracy, however, must continually diminish, as will appear from the following table:

·3	is less than	·33,	which is less than	$\frac{1}{3}$,
·33	"	·333,	"	$\frac{1}{3}$,
·333	"	·3333,	"	$\frac{1}{3}$,

and so on, ad infinitum. The fractions ·3, ·33, ·333, ·3333, &c., continually increase, and yet fall short of $\frac{1}{3}$; therefore the more figures we take, the nearer we approach to $\frac{1}{3}$; but as there is always a remainder, we never actually reach it.

Reduce $\frac{2}{7}$ to a decimal.

$$7\,)\,\underline{2\qquad}$$
$$\cdot\dot{2}8571\dot{4}$$

$\frac{2}{7} > \cdot 2$ by $\frac{3}{35}$ .... A

$\frac{2}{7} > \cdot 28$ by $\frac{1}{175}$, which reduces the error by .... $\frac{2}{25}$ ... B

$\frac{2}{7} > \cdot 285$ by $\frac{1}{1400}$, which further reduces the error by $\frac{1}{200}$ ... C

$\frac{2}{7} > \cdot 2857$ by $\frac{1}{70000}$, " " $\frac{7}{10000}$ ... D

$\frac{2}{7} > \cdot 28571$ by $\frac{3}{700000}$, " " $\frac{1}{100000}$ ... E

$\frac{2}{7} > \cdot 285714$ by $\frac{1}{3500000}$, " " $\frac{1}{250000}$ ... F

&c.

Estimate of error in different Units.

	£. s. d.	lb. av.	Cwt.	Ton.	Yd.	Mile.	7920 miles, Earth's diameter.	93500000 miles, Sun's distance.
A	1s. $8\frac{4}{7}d.$	$1\frac{13}{35}$ oz.	$9\frac{3}{5}$ lb.	1 cwt. 2 qrs. 24 lb.	$3\frac{3}{35}$ in.	$150\frac{6}{7}$ yd.	$678\frac{6}{7}$ m.	$8014285\frac{5}{7}$ miles.
B	$1\frac{13}{35}d.$	$1\frac{81}{175}$ dr.	$10\frac{6}{25}$ oz.	$12\frac{4}{5}$ lb.	$\frac{36}{175}$ in.	$10\frac{2}{35}$ yd.	$45\frac{9}{35}$ m.	$534285\frac{5}{7}$ "
C	$\frac{24}{35}$ f.	5 gr.	$1\frac{7}{25}$ oz.	$1\frac{3}{5}$ lb.	$\frac{9}{350}$ in.	$1\frac{9}{35}$ yd.	$5\frac{23}{35}$ m.	$66785\frac{5}{7}$ "
D	$\frac{12}{875}$ f.	$\frac{1}{10}$ gr.	$11\frac{1}{5}$ gr.	$8\frac{24}{125}$ dr.	$\frac{9}{17500}$ in.	$\frac{792}{875}$ in.	$199\frac{23}{175}$ yd.	$1335\frac{5}{7}$ "
E	$\frac{18}{4375}$ f.	$\frac{3}{100}$ gr.	$3\frac{9}{25}$ gr.	$2\frac{286}{625}$ dr.	$\frac{27}{175000}$ in.	$\frac{1188}{4375}$ in.	$59\frac{647}{875}$ yd.	$400\frac{5}{7}$ "
F	$\frac{6}{21875}$ f.	$\frac{1}{500}$ gr.	$\frac{28}{125}$ gr.	$4\frac{12}{25}$ gr.	$\frac{9}{875000}$ in.	$\frac{396}{21875}$ in.	$3\frac{4299}{4375}$ yd.	$26\frac{5}{7}$ "
		&c.			&c.		&c.	

This table may be read thus: $\frac{2}{7}$ of a unit exceeds ·2 of it by $\frac{3}{35}$ of it. If this unit be £1, this error = 1s. $8\frac{4}{7}d.$; if 1 lb., the error = $1\frac{13}{35}$ oz., &c. In line B, this error is only $\frac{1}{175}$ of £1 or of 1 lb., &c.

Exercise XV.

(1) Construct a similar table with $\frac{7}{11}$ and with $\frac{5}{13}$.

(2) Estimate the difference between :

a. $\frac{5}{17}$ of 1 mile and ·2941 of 1 mile.

b. £$\frac{13}{19}$ and £·6842.

Examining this table, we find (*a*) that the error continually diminishes; (*b*) that the larger the unit handled, the further must the process be carried to render the error insignificant. Thus the lb. column shews a microscopic error in the 4th line, whilst with the tons the error in this line is still appreciable. The same fact is yet more strikingly apparent if we compare the yard column with that of the sun's distance.

The continually diminishing error can be made as small as we please; that is to say, it can be made less than any assigned quantity, however small. Thus,

$\frac{2}{7}$ > ·2 and < ·3 ; it lies between ·2 and ·3, and does not differ from either by ·1.
$\frac{2}{7}$ > ·28 and < ·29 ; ,, ·28 and ·29, ,, ·01.
$\frac{2}{7}$ > ·285 and < ·286 ; ,, ·285 and ·286, ,, ·001.
⋮
$\frac{2}{7}$ > ·285714 and < ·285715 ; ,, ·285714 and ·285715, ,, ·000001.

Suppose a problem is given to which the answer must not be wrong by so much as $\frac{1}{24780}$ of the unit; $\frac{1}{24780}$ > ·00001. If the answer be a decimal correct to 5 places, its error < ·00001, and the required degree of accuracy is attained.

The series

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\ldots\ldots$$

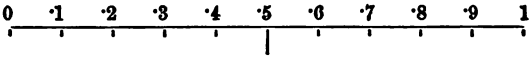
$$\frac{285714}{1000000}+\frac{285714}{1000000000000}+\frac{285714}{1000000000000000000}+\ldots\ldots$$

respectively approach $\frac{1}{3}$ and $\frac{2}{7}$, to which, by taking a sufficient number of decimal places, they may be made as near as we please. $\frac{1}{3}$ and $\frac{2}{7}$ are called the Limits of these series.

"The word limit implies a fixed magnitude, to which another and a variable magnitude may be made as nearly equal as we please, it being impossible, however, that the variable magnitude can absolutely attain or be equal to the fixed magnitude. In this strict sense of the word, there are two conditions which must be fulfilled before A can be called the limit of P : first, P must never become equal to

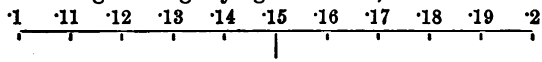
A; secondly, P must be capable of being made as nearly equal to A as we please."*

Since this limit can never be reached, we must stop somewhere near it, and we have to settle the principle on which the stoppage is to be regulated. If the distance between 0 and 1 be divided decimally, we shall find the following stations marked out along the line:



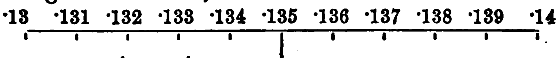
A

Again: the distance between, say ·1 and ·2, may be subdivided, which, under a glass magnifying it ten-fold, would look thus:



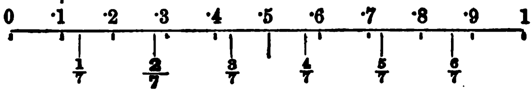
B

Similarly, any one of these intervals, say ·13 and ·14, might be further magnified ten-fold, and would look thus:



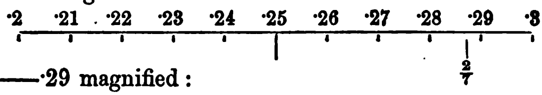
C

Now take $\frac{2}{7} = \cdot\dot{2}8571\dot{4}$



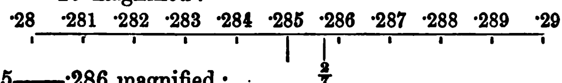
D

·2——·3 magnified:



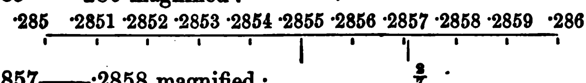
E

·28——·29 magnified:



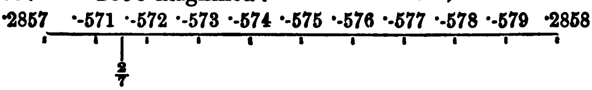
F

·285——·286 magnified:



G

·2857——·2858 magnified:



H

And so on. This magnifying of successively smaller intervals, which we may imagine carried on ad infinitum, brings home to us the idea of the infinite divisibility of a line; for, however small the

* Penny Cycl., art. LIMIT.

intervals are made, or, in other words, however numerous the points to which names are given, there must always be between them an infinitely larger number of points left unnamed. Professor De Morgan, in one of his lectures, has compared Arithmetic to the piano and Geometry to the violin: the former instrument by its structure can only give name to lengths of string at certain intervals; the latter omits no point along the whole line.

We see now that *midway* between	0	and 1	lies ·5
„	·1	·2	·15
„	·13	·14	·135
„	·2	·3	·25
„	·28	·29	·285
„	·285	·286	·2855
„	·2857	·2858	·28575, &c.

Line A shews that all quantities below ·5 are nearer to 0 than to 1; ·5 is midway, and those above ·5 are nearer to 1. $\frac{2}{7}$ is nearer to ·3 than to ·2, being above ·25; $\frac{2}{7}$ is nearer to ·29 than to ·28, being above ·285; $\frac{2}{7}$ is nearer to ·286 than to ·285, being above ·2855; $\frac{2}{7}$ is nearer to ·2857 than to ·2858, being below ·28575, and so on. If we wish to express $\frac{2}{7}$ as a decimal to

one place,	we shall be nearest the truth	by calling it	·3
two places,	„	„	·29
three	„	„	·286
four	„	„	·2857
five	„	„	·28571
six	„	„	·285714
seven	„	„	·2857143, &c.

Rule: *In* CURTAILING *a decimal, if the first figure rejected be less than* 5, *make no change in the last figure retained; if more than* 4 *increase by* 1 *the last figure retained.*

If the first figure rejected be 5, it is immaterial whether we adopt the higher or the lower value if the fraction terminates at the 5; but as there generally are more figures after the 5, the higher value is somewhat more correct.

EXERCISE XVI.

(1) Curtail to 5 places $\cdot430718$, $\cdot010\dot{2}0\dot{3}$, $\cdot430\dot{7}9\dot{8}$.
(2) „ to 4 places $\cdot\dot{6}$, $\cdot\dot{4}$, $\cdot\dot{2}\dot{7}$, $\cdot2\dot{7}$, $4\cdot\dot{0}\dot{9}$, $4\cdot0\dot{9}$.
(3) „ to 7 places $\cdot\dot{9}$, $\cdot\dot{0}\dot{9}$, $\cdot000\dot{9}$, $\cdot37\dot{9}$, $\cdot42\dot{9}$, $\cdot35\dot{9}$.
(4) „ $999\cdot\dot{9}$ to nearest integer.
(5) Reduce $\frac{1}{17}$ to a decimal to 1, 2, 3......16 places.

§ 3. ADDITION AND SUBTRACTION.

Work the following by vulgar fractions and by decimals, correct to 4 places:

$$7\tfrac{2}{3} + 13\tfrac{3}{7} + 5\tfrac{5}{14} + 43\tfrac{7}{15} + \tfrac{11}{18} + 6\tfrac{4}{9} + 10\tfrac{6}{35} + 100\tfrac{8}{13} + 3\tfrac{9}{91} + 1\tfrac{12}{65}.$$

	8190	
7	2730	5460
13	1170	3510
5	585	2925
43	546	3822
	455	5005
6	910	3640
10	234	1404
100	630	5040
3	90	810
1	126	1512
192	8190) 33128 (4	

$$\frac{368}{8190} = \frac{184}{4095}$$

4095)184·00(·04493
20200
38200
13450
&c.

$7\frac{2}{3}$ =	7·66667	6
$13\frac{3}{7}$ =	13·42857	1
$5\frac{5}{14}$ =	5·35714	2
$43\frac{7}{15}$ =	43·46667	6
$\frac{11}{18}$ =	·61111	1
$6\frac{4}{9}$ =	6·44444	4
$10\frac{6}{35}$ =	10·17143	8
$100\frac{8}{13}$ =	100·61538	4
$3\frac{9}{91}$ =	3·09890	1
$1\frac{12}{65}$ =	1·18462	5
	192·04493	

Ans. 192·0449.

N.B. To insure the correctness of the last place required (here the fourth), it is well to carry on the fractions *to one place more* than is specified.

EXERCISE XVII.

Simplify the following by vulgar fractions and by decima

(1) $\frac{13}{15} + 6\frac{8}{15} + 9\frac{11}{20} + 100\frac{13}{30}$ to 5 places.

(2) $10\frac{4}{11} + 9\frac{5}{13} + 6\frac{2}{3} + 8\frac{112}{143} + 3\frac{5}{9}$ to 5 places.

(3) $8\frac{2}{3} + 9\frac{5}{7} + 3\frac{11}{21} + 6\frac{23}{77} + 4\frac{5}{6} + 10\frac{13}{33} + 2\frac{7}{9}$ to 5 places.

(4) $8\frac{1}{2} + \frac{5}{6} + \frac{11}{14} + 7\frac{3}{10} + 14\frac{17}{21} + \frac{11}{15} + \frac{12}{35}$ to 5 places.

(5) $8\frac{3}{5} + 7\frac{5}{8} + 2\frac{4}{7} + 9\frac{13}{20} + 11\frac{4}{35} + 10\frac{15}{56} + 12\frac{33}{70}$ to 5 places.

(6) $\frac{2}{7}$ of $18 + \frac{3}{5}$ of $1\frac{4}{21}$ to 4 places.

(7) $\frac{13}{15} - \frac{8}{21}$ to 5 places.

(8) $13\frac{2}{75} - 3\frac{8}{15}$ to 5 places.

(9) $8\frac{4}{15} - 7\frac{11}{25}$ to 5 places.

(10) $7\frac{4}{7} - \frac{1}{2}$ of $8\frac{3}{11}$ to 5 places.

(11) $\frac{2}{3}$ of $6\frac{1}{2} - \frac{2}{5}$ of 4 to 5 places.

(12) $5\frac{1}{3}$ of $4\frac{1}{2} - 3\frac{1}{4}$ of $3\frac{1}{5}$ to 5 places.

(13) $\frac{1}{3} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17}$ to 5 places.

By decimals only:

(14) $\frac{5}{19} + \frac{8}{17} + \frac{4}{23} + \frac{11}{29}$ to 7 places.

(15) $\frac{14}{17} - \frac{7}{12}$ to 7 places.

(16) $(\frac{31}{41} + \frac{14}{43}) - (\frac{31}{41} - \frac{14}{43})$ to 7 places.

(17) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}$ to 4 places.

(18) $(\frac{17}{59} + \frac{14}{61}) + (\frac{17}{59} - \frac{14}{61})$ to 4 places.

(19) $\cdot 4\dot{1}\dot{7} + 4\cdot 31\dot{6}\dot{2} + 71\cdot\dot{5}\dot{8} + 4\cdot 3\dot{4}8\dot{7}$ to 3 places.

(20) $\cdot\dot{8}246\dot{1} + 43\cdot\dot{7}86\dot{2} - 17\cdot\dot{1}76\dot{4}$ to 5 places.

(21) $\frac{1}{3} + \frac{1}{3 \times 3} + \frac{1}{3 \times 3 \times 3} + \frac{1}{3 \times 3 \times 3 \times 3} + \ldots\ldots$ to 5 places.

(22) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\ldots$ to 6 places.

(23) $1 + 1 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{2 \times 3 \times 4 \times 5} + \ldots\ldots$ to 9 places.

(24) $16 \times (\frac{1}{5} - \frac{1}{3 \times 5^3} + \frac{1}{5 \times 5^5} - \frac{1}{7 \times 5^7} + \frac{1}{9 \times 5^9} - \frac{1}{11 \times 5^{11}} + \&c.) - \frac{4}{239}$ 5 places.

§ 4. Multiplication.

Case I. By a power of 10.

By vulgar fractions and by decimals, correct to 6 places, multiply $4\frac{5}{7}$ by 10, 100 and 10000.

$4\frac{5}{7}\times 10=47\frac{1}{7}$	$4\frac{5}{7}=4\cdot714285714\ldots\ldots$
$=47\cdot142857$	$4\frac{5}{7}\times 10=47\cdot142857$
$4\frac{5}{7}\times 100=471\frac{3}{7}$	$4\frac{5}{7}=4\cdot714285714\ldots\ldots$
$=471\cdot428571$	$4\frac{5}{7}\times 100=471\cdot428571$
$4\frac{5}{7}\times 10000=47142\frac{6}{7}$	$4\frac{5}{7}=4\cdot71428571428\ \ldots\ldots$
$=47142\cdot85714\dot{3}$	$4\frac{5}{7}\times 10000=47142\cdot85714\dot{3}$

Exercise XVIII.

(1) $\cdot\dot{3}\times 10$, 1000, 100000, to 3 places.

(2) $4\cdot\dot{7}\dot{2}\times 10$, 100, 1000000, to 4 places.

By vulgar fractions and by decimals to 6 places:

(3) $5\frac{8}{9}\times 10$, 100, 10000.

(4) $\frac{7}{12}\times 10$, 100, 10000.

(5) $\frac{13}{14}\times 10000$.

(6) $\frac{5}{17}\times 100$.

(7) $\frac{1}{7200}\times 1000000$.

Case II. By any integer.

By vulgar fractions and by decimals, correct to 4 places, find $19\frac{5}{11}\times 7$.

By vulgar fractions:

$$19\tfrac{5}{11}\times 7=136\tfrac{2}{11}=136\cdot1818.$$

By decimals:

$$\begin{array}{r|l} 19\tfrac{5}{11}\times 7=19\cdot\dot{4}\dot{5}\times 7=19\cdot4545 & 4 \\ 19\cdot4545 & 4 \\ 19\cdot4545 & 4 \\ 19\cdot4545 & 4 \\ 19\cdot4545 & 4 \\ 19\cdot4545 & 4 \\ 19\cdot4545 & 4 \\ \hline 136\cdot181\cancel{7} & \cancel{8} \\ 8 & \end{array}$$

Ans. 136·1818.

If we had not multiplied the 4's to the right of the vertical line, our fourth place would have been greatly inaccurate, as we should

have lost the carriage. Even with this multiplication, the fourth place is inaccurate, unless we allow for the rejection of the high figure in the fifth place. We therefore proceed as follows:

$$\begin{array}{r} 19{\cdot}45454 \\ 7 \\ \hline 136{\cdot}1818 \end{array}$$

Wording: 28, carry *three;* 35, 38' carry 3, &c.

It is most important to remember that when we multiply the first figure rejected, we carry the "nearest ten." Thus to 21, 22, 23 and 24, 20 is the nearest ten; to 26, 27, 28 and 29, 30 is the nearest ten; for 25, it is better to carry 3 than 2, as figures following the first rejected might increase, but could not decrease, the 25. In other words, if the units' figure be above 4, carry the higher ten; if below 5, the lower.

Multiply $\frac{11}{19}$ by 600 to 5 places.

By vulgar fractions:

$$\tfrac{11}{19} \times 600 = \tfrac{6600}{19} = 347\tfrac{7}{19} = 347{\cdot}36842\dot{1}.$$

By decimals:

$$\tfrac{11}{19} \times 600 = {\cdot}5789473684 \times 600 = 57{\cdot}89473684 \times 6.$$

$$\begin{array}{r} 57{\cdot}89473684 \\ 6 \\ \hline 347{\cdot}36842 \end{array}$$

Hence, to multiply by any multiple of a power of 10, first multiply by the power of 10 by shifting the point. It will be observed that, in preparing the multiplicand, we must carry out the original decimal one figure more for every cipher. In the simplest form the sum would stand thus:

$$\tfrac{11}{19} \times 600 = {\cdot}57894737 \times 600.$$

$$\begin{array}{r} 57{\cdot}894737 \\ 6 \\ \hline 347{\cdot}36842 \end{array}$$

Exercise XIX.

By vulgar fractions and by decimals:

(1) $\frac{8}{13} \times 7000$ to 4 places.

(2) $\frac{5}{21} \times 300$,,

(3) $\frac{7}{11} \times 80000$,,

(4) $\frac{4}{19} \times 19000$,,

(5) $\frac{5}{18} \times 90000$ to 4 places.
(6) $53\frac{1}{120} \times 50$ „
(7) $\frac{30}{37} \times 2000$ „
(8) $\frac{41}{42} \times 90000$ „
(9) $3\frac{8}{11} \times 6000000$ „

By decimals only:

(10) $3{\cdot}4\dot{3}\dot{1} \times 500$ to 4 places.
(11) $416{\cdot}7\dot{1} \times 4000$ „
(12) $\cdot\dot{7}8\dot{4} \times 30000$ „

Multiply $\frac{45}{73}$ by 427 to 6 places.

By vulgar fractions:

$$\frac{45}{73} \times 427 = \frac{19215}{73} = 263{\cdot}219178.$$

By decimals:

$$\frac{45}{73} \times 427 = \cdot616438356164\ldots\ldots \times 427.$$

$$\text{For } 400 \left\{ \begin{array}{r} \cdot616438356\cancel{164} \\ 4 \\ \hline 246{\cdot}575342 \end{array} \right.$$

$$\text{For } 20 \left\{ \begin{array}{r} \cdot616438356 \\ 2 \\ \hline 12{\cdot}328767 \end{array} \right. \qquad \begin{array}{r} 246{\cdot}575342 \\ 12{\cdot}328767 \\ 4{\cdot}315068 \\ \hline 263{\cdot}219177 \end{array}$$

$$\text{For } 7 \left\{ \begin{array}{r} \cdot616438356 \\ 7 \\ \hline 4{\cdot}315068 \end{array} \right.$$

This process may be contracted thus:

$$\begin{array}{r} \cdot616438356 \\ 724 \\ \hline 246{\cdot}575342 \\ 12{\cdot}328767 \\ 4{\cdot}315068 \\ \hline 263{\cdot}219177 \end{array}$$

Notice that the figures of the multiplier are reversed, each figure of the multiplier being written beneath that decimal place of the multiplicand which will give the last decimal place required. It will also be seen that the last figure was not accurate, being 7 instead of 8. Had the question been worked to seven places, the sixth would have been accurate.

EXERCISE XX.

By vulgar fractions and by decimals :

(1) $43\frac{40}{41} \times 259$ to 4 places. (4) $\frac{14}{15} \times 3015$ to 4 places.
(2) $\frac{16}{29} \times 1043$,, (5) $10\frac{11}{19} \times 9500$,,
(3) $1\frac{4}{11} \times 3020$,, (6) $13\frac{148}{2048} \times 8432$,,

CASE III. By a fraction.

Multiply $\frac{4}{23}$ by $\frac{2}{5}$ to 4 places.

By vulgar fractions :

$$\frac{4}{23} \times \frac{2}{5} = \frac{8}{115} = \cdot 069565. \qquad \textit{Ans.}\ \cdot 0696.$$
$$6$$

By decimals :

$$\frac{4}{23} \times \frac{2}{5} = \cdot 173913 \ldots\ldots \times \cdot 4 = \cdot 0173913 \times 4.$$

$$\begin{array}{r} \cdot 01739 \\ 4 \\ \hline \cdot 0696 \end{array} \qquad \textit{Ans.}\ \cdot 0696.$$

Multiply $\frac{14}{15}$ by $\frac{1}{125}$ to 6 places.

By vulgar fractions :

$$\frac{14}{15} \times \frac{1}{125} = \frac{14}{1875} = \cdot 0074\dot{6}. \qquad \textit{Ans.}\ \cdot 007467.$$

By decimals :

$$\frac{14}{15} \times \frac{1}{125} = \cdot 9\dot{3} \times \cdot 008 = \cdot 0009\dot{3} \times 8.$$

$$\begin{array}{r} \cdot 0009333 \\ 8 \\ \hline \cdot 007466 \end{array} \qquad \textit{Ans.}\ \cdot 007466.$$

Multiply $542\frac{13}{88}$ by $314\frac{15}{16}$ to 4 places.

By vulgar fractions :

$$542\tfrac{13}{88} \times 314\tfrac{15}{16} = \frac{47709}{88} \times \frac{5039}{16} = \frac{240405651}{1408} = 170742\tfrac{915}{1408} = 170742 \cdot 64985.$$
$$9$$

Ans. 170742·6499.

By decimals :

$$542\tfrac{13}{88} \times 314\tfrac{15}{16} = 542 \cdot 147\dot{7}\dot{2} \times 314 \cdot 9375.$$

$$\text{For } 314 \left\{ \begin{array}{r} 542 \cdot 1477272 \\ 413 \\ \hline 162644 \cdot 3182 \\ 5421 \cdot 4773 \\ 2168 \cdot 5909 \\ \hline 170234 \cdot 3864 \end{array} \right.$$

$$\text{For } \cdot 9 \left\{ \begin{array}{r} 542 \cdot 1477272 \\ 9 \\ \hline 487 \cdot 9329 \end{array} \right.$$

For ·03 $\left\{\begin{array}{r} 542\cdot1477272 \\ 3 \\ \hline 16\cdot2644 \end{array}\right.$

For ·007 $\left\{\begin{array}{r} 542\cdot1477272 \\ 7 \\ \hline 3\cdot7950 \end{array}\right.$

For ·0005 $\left\{\begin{array}{r} 542\cdot1477272 \\ 5 \\ \hline \cdot2711 \end{array}\right.$

Sum of the products.

$$\begin{array}{r} 170234\cdot3864 \\ 487\cdot9329 \\ 16\cdot2644 \\ 3\cdot7950 \\ \cdot2711 \\ \hline 170742\cdot6498 \end{array}$$

This process may be contracted thus:

$$\begin{array}{r} 542\cdot1477272 \\ 5739413 \\ \hline 1626443182 \\ 54214773 \\ 21685909 \\ 4879329 \\ 162644 \\ 37950 \\ 2711 \\ \hline 170742\cdot6498 \end{array}$$

Notice that the fractional part of the multiplier is reversed as well as the integral, which we might have anticipated. This method of multiplication is useful even with non-recurring decimals.

Multiply 43·7842195 by 2·102736 to 4 places.

$$\begin{array}{r} 43\cdot78422 \\ 6372012 \\ \hline 875684 \\ 43784 \\ 876 \\ 306 \\ 13 \\ 2 \\ \hline 92\cdot0665 \end{array}$$

It now only remains to shew how to arrange the terms of the multiplication.

Rule I. In contracted Multiplication of Decimals, place the unit of the multiplier under the last place of decimals to be retained,

then reverse the multiplier, and if possible* make the multiplier overlap the multiplicand by one figure to the left, and the multiplicand overlap the multiplier by one figure to the right.

II. Multiply each figure of the multiplier by the figure next to the right above it; do not write down this result, but merely carry the "nearest ten," and proceed to multiply as usual.

III. Write the products under one another, placing the first figures retained in a vertical line.

IV. Add the several products, and mark off, counting from the right of their sum, the number of decimal places arranged for.

Multiply 43·7842195 by 2·102736.

43784	2195
210	2736
87568	4390
4378	42195
87	5684390
30	64895365
1	313526585
	2627053170
92·066	6545745520

Suppose that in this case only three decimal places were required, then all the work to the right of the vertical line would be wasted. How can this useless work be avoided?

Examine each separate product. The 8 to the immediate left of the vertical in the first line is obtained by multiplying 4 in the multiplicand by 2 in the multiplier; write down the multiplicand, putting this 2 under the 4, thus:

$$\begin{array}{r} 43{\cdot}7842195 \\ \underline{2} \\ 87{\cdot}568 \end{array}$$

The 8 of the second line is obtained by multiplying 1 by 8. Again, writing the 1 under the 8, we get,

$$\begin{array}{r} 43{\cdot}7842195 \\ \underline{12} \\ 87{\cdot}568 \\ 4{\cdot}378 \end{array}$$

* This can always be done by putting or supposing ciphers.

The 7 of the third line is obtained by multiplying 2 by 3, and carrying 1 from the 2 × 7. Writing the 2 under the 7, we get,

43·7842195
2 012
87·568
4·378
87

The 0 of the next line is obtained by multiplying 7 by 4, and carrying 2 from 7 × 3. Similarly, the 1 in the next line is obtained by carrying from 3 × 4. The multiplication by 6 yielded no result in the first three places of decimals, and is therefore omitted. Arranging now the whole sum, we have,

	43·7842195		43·7842
	6372 012		472 012
	87568		87568
A	4378	B	4378
	87		87
	30		30
	1		2
	92·064		92·065

In sum A, the third place is very inaccurate, because we have lost the carriage in the addition. This is partly compensated for by observing the law for curtailing decimals given in § 2, as in sum B, and even then the result is inaccurate. If the third place be of importance, arrange for four places.

5·790684 × ·023056 to 5 places.

Either,	or,
5·790684	·023056
65 0320Q*	70975
11581	11528
1737	1614
29	207
3	1
·13350Q	·1335Q

Ans. ·1335.

* The Q indicates the absent units' figure.

$634{\cdot}25\dot{7}6\dot{9} \times 12{\cdot}0\dot{6}$ to 7 places, with *exactness*; we therefore shall take 8 places.

Either,	or,
634·2576976977	12·06666666667
7666 666666021	779 6796752436
634257697698	724000000000
126851539539	36200000000
3805546186	4826666666
380554619	241333333
38055462	60333333
3805546	8446666
380555	724000
38055	108600
3806	8446
381	724
38	108
4	8
7653·37621889	1
	7653·37621885

Ans. 7653·3762189.

Note first the great discrepancy between the two eighth places, which, however, does not affect the seventh. This discrepancy is due to the fact that in the left-hand sum the number of over-estimates considerably exceeds that of the under-estimates, whilst in the right-hand sum the reverse is the case.

Secondly, in the left-hand sum the fourth and succeeding lines are derived by successively curtailing the third. Care must be taken not to derive any of these lines from its immediate predecessor, as this may cause an accumulation of inaccuracy.

A computer should at all times test his results by some rough estimate, and this is especially important in decimals, where a displacement of the point produces so vital a difference. The most fertile source of error in contracted multiplication is an erroneous arrangement of the factors, but such an error may be easily detected.

In the last example of the preceding page, 5 and a fraction is to be multiplied by ·02... ; the product must therefore be something over ·1, as $5 \times {\cdot}02 = {\cdot}1$. In the next example, upwards of 600 is taken more than 12 times; therefore the product must be more than 7200.

EXERCISE XXI. (*a*).

By vulgar fractions and by decimals, correct to 5 places, find :

(1) $32\frac{5}{7} \times 1\frac{4}{11}$.

(2) $7\frac{2}{3} \times \frac{5}{13}$.

(3) $1\frac{4}{15} \times \frac{7}{8}$.

(4) $1\frac{1}{640} \times \frac{1}{1024}$.

(5) $6528\frac{1}{25000} \times 3794\frac{1}{128}$.

(6) $\frac{11}{19000} \times \frac{4}{365}$.

By decimals only, working each question in two ways, by making each factor multiplier and multiplicand in turn :

(7) $\cdot 03\dot{7}9\dot{4} \times 5\cdot 0\dot{0}8\dot{4}$ to 5 places.

(8) $\cdot 111\dot{6} \times 43\cdot 74\dot{2}85\dot{3}$ to 8 places.

(9) $\cdot 9321875 \times 4\cdot 2688$ to 4 places.

(10) $\cdot 38 \times \cdot 04125$ to 3 places.

(11) $1\cdot 48279 \times \cdot\dot{3}$ to 5 places.

(12) $468\cdot\dot{1}\dot{2} \times 299\cdot 875$ to the nearest integer.

(13) The circumference of a circle is 3·14159265...times its diameter. Find in miles to the nearest integer the circumference of the earth, reckoning the equatorial diameter $7925\frac{7}{8}$ miles.

(14) Find in miles to the nearest integer the polar diameter, which is $\frac{299}{300}$ of the equatorial diameter.

(15) Find the diameter of the sun, which is 112·06...times that of the earth, correct to within 100 miles.

§ 5. NUMBER OF DECIMAL PLACES TO BE RETAINED IN FINDING A CONTINUED PRODUCT.

The factors of a continued product may be proper or improper fractions, and our *modus operandi*, in finding an approximate product with a given limit of error, must depend on their nature in this respect.

$2\frac{7}{8} \times 4719\frac{14}{15} \times 6\frac{4}{11} \times \frac{1}{120}$.

By vulgar fractions : $\frac{23}{8} \times \frac{70799}{15} \times \frac{7\cancel{0}}{11} \times \frac{1}{12\cancel{0}} = \frac{11398639}{15840}$.

70799	$1584\cdot\cancel{0}$) $1139863\cdot9$ ($719\cdot611$
161	3106
70799	15223
424794	9679
70799	1750
11398639	1660
	76

Ans. $719\frac{9679}{15840}$, or to three places $719\cdot611$.

By decimals to three places: $2\cdot875 \times 4719\cdot9\dot{3} \times 6\cdot\dot{3}\dot{6} \times \cdot008\dot{3}$.

It would seem that the factors might be taken in any order; let us take them from right to left.

6·4	4719·9333
38 $\cancel{0}\cancel{0}$	3 50
51	236997
2	14160
·053 first product.	250·157 second product.
	5782
	500314
	200126
	17511
	1251
	719·202 third product.

This result is greatly inaccurate even in the first place of decimals. The first product ·053 is accurate to the third place, but is nevertheless inaccurate for want of the succeeding places. Now this inaccuracy is multiplied by nearly 4720 in finding the second product, and this again is nearly trebled in finding the last product. These accumulated inaccuracies, only slightly compensated for by occasional over-estimates, have given as final result the very appreciable error of ·409.

These errors can be avoided thus: Arrange the factors in descending order, and move the several points so as to make each factor but the first a proper fraction. We may then, to ensure accuracy, keep one more than the assigned number of places all through, and each successive multiplication by a proper fraction will reduce the error.

Thus the problem $4719{\cdot}9\dot{3} \times 6{\cdot}\dot{3}\dot{6} \times 2{\cdot}875 \times {\cdot}008\dot{3}$,

becomes $47199{\cdot}\dot{3} \times {\cdot}\dot{6}\dot{3} \times 2{\cdot}875 \times {\cdot}008\dot{3}$

$= 471993{\cdot}\dot{3} \times {\cdot}\dot{6}\dot{3} \times {\cdot}2875 \times {\cdot}008\dot{3}$,

or better still, $4719{\cdot}9\dot{3} \times {\cdot}8\dot{3} \times {\cdot}\dot{6}\dot{3} \times {\cdot}2875$.

$$
\begin{array}{r}
4719{\cdot}9333 \\
\underline{83333\ 338} \\
37759466 \\
1415980 \\
141598 \\
14160 \\
1416 \\
142 \\
14 \\
\underline{1} \\
3933{\cdot}2777 \\
\underline{36363\ 636} \\
23599666 \\
1179983 \\
235997 \\
11800 \\
2360 \\
118 \\
24 \\
\underline{1} \\
2502{\cdot}9949 \\
\underline{5\ 782} \\
5005990 \\
2002395 \\
175209 \\
\underline{12515} \\
719{\cdot}6109 \\
1
\end{array}
$$

Ans. 719·611.

Exercise XXI. (*b*).

Find continued product of:

(1) $49{\cdot}58\dot{3} \times 11{\cdot}4\dot{3}0\dot{2} \times {\cdot}01$ to 4 places.

(2) $\frac{18}{59} \times .0\dot{1}0\dot{7} \times \frac{12}{65} \times {\cdot}04$ to 6 places.

(3) $2857{\cdot}\dot{5}\dot{4} \times 7\frac{58}{91} \times 12\frac{7}{11} \times \frac{1}{6250}$ to 3 places.

(4) $17{\cdot}\dot{8}1\dot{2} \times 5000\frac{14}{15} \times \frac{11}{16} \times {\cdot}001$ to 4 places.

(5) ${\cdot}0\dot{5}9\dot{3} \times {\cdot}0\dot{5}9\dot{3} \times {\cdot}0\dot{5}9\dot{3}$ to 8 places.

(6) ${\cdot}0\dot{8}0\dot{5} \times {\cdot}0\dot{8}0\dot{5} \times {\cdot}0\dot{8}0\dot{5} \times .0\dot{8}0\dot{5} \times {\cdot}0\dot{8}0\dot{5} \times {\cdot}0\dot{8}0\dot{5}$ to 9 places.

§ 6. Division.

Case I. By a power of 10. Shift the decimal point as many places to the left as there are ciphers in the divisor, prefixing ciphers if necessary, and curtail the decimal thus obtained.

$\frac{5}{13} \div 100$ to 5 places.

$\frac{5}{13} \div 100 = {\cdot}38461\ldots \div 100 = {\cdot}0038\cancel{4}61.$ (5 written under the struck 4) *Ans.* ·00385.

Case II. By a divisor with few significant figures.

$40{\cdot}12\dot{8}6\dot{3} \div 70000$ to 5 places.

$40{\cdot}12\dot{8}6\dot{3} \div 70000 = {\cdot}004012\dot{8}6\dot{3} \div 7.$

$$\begin{array}{r} 3 \\ 7\,)\,{\cdot}00401\cancel{2}\cancel{8} \\ \hline {\cdot}000573 \end{array}$$

Ans. ·00057.

$395{\cdot}\dot{4}\dot{0} \div 56000$ to 6 places.

$395{\cdot}\dot{4}\dot{0} \div 56000 = {\cdot}395\dot{4}\dot{0} \div 56.$

$$\begin{array}{r} 8\,)\,{\cdot}395404 \\ \hline 7\,)\,{\cdot}049426 \\ \hline {\cdot}007061 \end{array}$$

Ans. ·007061.

$6 \div 51700$ to 7 places.

$6 \div 51700 = {\cdot}06 \div 517.$

$$\begin{array}{l} 517)\,{\cdot}0600(\,{\cdot}000116\cancel{0}\cancel{5} \\ \quad\quad\;\; 830 \quad\quad 1 \\ \quad\quad\;\; 3130 \\ \quad\quad\quad 2800 \end{array}$$

Ans. ·0001161.

${\cdot}\dot{6} \div 8213000$ to 12 places.

${\cdot}\dot{6} \div 8213000 = {\cdot}000\dot{6} \div 8213.$

$$\begin{array}{l} 8213)\,{\cdot}00066666(\,{\cdot}000000081172\cancel{1} \\ \quad\quad\quad\; 9626 \\ \quad\quad\quad\; 14136 \\ \quad\quad\quad\;\; 59236 \\ \quad\quad\quad\;\;\; 17456 \\ \quad\quad\quad\;\;\;\; 10306 \end{array}$$

Ans. ·000000081172.

$5{\cdot}\dot{1}4285\dot{7} \div {\cdot}613$ to 8 places.

$5{\cdot}\dot{1}4285\dot{7} \div {\cdot}613 = 5142{\cdot}\dot{8}5714\dot{2} \div 613.$

$$\begin{array}{l} 613)\,5142{\cdot}\dot{8}5714\dot{2}(8{\cdot}38965276\cancel{1} \\ \quad\quad\; 2388 \\ \quad\quad\;\; 5495 \\ \quad\quad\;\;\; 5917 \\ \quad\quad\;\;\;\; 4001 \\ \quad\quad\;\;\;\;\; 3234 \\ \quad\quad\;\;\;\;\;\; 1692 \\ \quad\quad\;\;\;\;\;\;\; 4668 \\ \quad\quad\;\;\;\;\;\;\;\; 3775 \\ \quad\quad\;\;\;\;\;\;\;\;\; 97 \end{array}$$

Ans. 8·38965276.

EXERCISE XXII.

(1) $743{\cdot}5\dot{8}\dot{7} \div 10$, 1000, 1000000, to 8 places.

(2) $8{\cdot}3\dot{7} \div 40$ to 5 places.

(3) $74 \div 35000$ to 6 places.

(4) ${\cdot}\dot{5}\dot{3} \div 30$ to 8 places.

(5) ${\cdot}\dot{7}2\dot{5} \div 4{\cdot}31$ to 5 places.

(6) ${\cdot}\dot{1}\dot{8} \div {\cdot}0007$ to 4 places.

(7) $47{\cdot}3\dot{4}\dot{5} \div {\cdot}01$, ${\cdot}001$, ${\cdot}00001$, to 5 places.

(8) ${\cdot}05 \div 630$ to 10 places.

(9) ${\cdot}28473 \div {\cdot}00761$ to 4 places.

(10) ${\cdot}\dot{5}8\dot{1} \div {\cdot}0009$ to 5 places.

(11) ${\cdot}\dot{3}7\dot{0} \div {\cdot}028$ to 6 places.

(12) $2{\cdot}\dot{5} \div 2{\cdot}5$ to 5 places.

(13) ${\cdot}001 \div 44$ to 6 places.

(14) $6{\cdot}587 \div 19$, 1900, to 7 places.

(15) $6{\cdot}\dot{5}8\dot{7} \div {\cdot}19$, $1{\cdot}9$, 19, to 7 places.

(16) $6{\cdot}5\dot{8}\dot{7} \div 4{\cdot}35$, 8100, to 9 places.

(17) $6{\cdot}58\dot{7} \div 4{\cdot}35$, 8100, 4350, ${\cdot}81$, to 10 places.

(18) ${\cdot}\dot{5}3846\dot{1} \div 1{\cdot}86$ to 3 places.

(19) ${\cdot}\dot{5}3846\dot{1} \div 36$, 360, ${\cdot}0036$, to 6 places.

(20) ${\cdot}\dot{5}3846\dot{1} \div 45$, $4{\cdot}5$, ${\cdot}45$, 45000, to 8 places.

(21) ${\cdot}\dot{0}\dot{7} \div 48007{\cdot}8$ to 5 places.

CASE III. By a divisor with many significant figures.

Lemma* 1. By adding a figure to the right of any series of digits, two operations are performed; the series is multiplied by 10, and is increased by the number of units expressed by that figure; e.g. $48579 = (4857 \times 10) + 9$. Conversely, by cutting off a figure from the right of any series of digits, two operations are performed; the series is diminished by the number of units expressed by that figure, and is divided by 10; e.g. $4857 = (48579 - 9) \div 10$.

* A Lemma is a proposition which is only used as subservient to the proof of another proposition.—De Morgan's Algebra.

Lemma 2. In a long series of digits, it matters comparatively little what the particular figure added or cut off happens to be, since the multiplication and division by 10 are respectively of *much* more moment than the addition or subtraction of a quantity less than 10; e.g. calling 480936 480930, is a much less error, as compared with the quantities under consideration, than calling 48 40 would be. 48 exceeds 40 by $\frac{1}{5}$ of 40; but 480936 exceeds 480930 by $\frac{1}{80155}$ of 480930. If from a series of digits we successively strike off the right-hand figures, the importance of the subtraction continually increases.

It follows directly from what was said in page 117, that the same effect is produced on the quotient by dividing the divisor as by multiplying the dividend. If, then, instead of successively "bringing down," i.e. adding figures to the right of the dividend, we cut off figures from the right of the divisor, the quotient will remain the same so long as the divisor has a large number of figures. On this truth, in fact, depends the method of guessing the figures of the quotient in integral division.

$11\cdot4\dot{2}8\dot{5} \div 3\cdot1415927.$

Full form.	Contracted form.
31415927)114285285 | (3·63781356	31415927)114285285(3·6378136
20037504 | 2 6	20037504
1187948 | 08	1187948
245470 | 275	245470
A 25558 | 7862	B 25559
426 | 04468	427
111 | 885415	113
17 | 6376342	19
1 | 92967078, &c.	—

The quotient in A is obtained by bringing down or adding figures to the successive remainders; that in B by cutting off figures from the divisor, multiplying, however, each time the figure just cut off by the new figure of the quotient to ascertain the carriage, which is always to be the nearest ten.

In form B, though the quotient remains accurate till the sixth place, the remainders (which are the preparation for future figures in the quotient) shew inaccuracies much earlier.

The last figure of the quotient in the case here given is unusually

accurate. It is advisable not to begin to cut off till the digits in the divisor are two more than the number of figures of the quotient still required.

Example: $\cdot 004\dot{2}3\dot{9} \div \cdot 3278$ to 7 places.

$$
\begin{array}{ll}
 & 32\cancel{7}\cancel{8})42\cdot 39\,(\cdot 0129324 \\
 & \quad\quad\;\; 9\,612 \\
 & \quad\quad\;\; 3\,0563 \\
\text{C} & \quad\quad\quad 10619 \\
 & \quad\quad\quad\quad 785 \\
 & \quad\quad\quad\quad 129 \\
 & \quad\quad\quad\quad\; \text{—}
\end{array}
$$

Ans. ·0129324.

$\cdot 004\dot{2}3\dot{9} \div \dot{3}27\dot{8}$ to 7 places.

$$
\begin{array}{ll}
 & 32\cancel{7}\cancel{8}\cancel{3}\cancel{2}\cancel{7}\cancel{8}\cancel{3})4239239\,(\cdot 0129311 \\
 & \quad\quad\quad\quad\quad 960911 \\
 & \quad\quad\quad\quad\quad 305246 \\
\text{D} & \quad\quad\quad\quad\quad\; 10197 \\
 & \quad\quad\quad\quad\quad\quad 362 \\
 & \quad\quad\quad\quad\quad\quad\; 34 \\
 & \quad\quad\quad\quad\quad\quad\quad 1
\end{array}
$$

Ans. ·0129311.

In C, only two figures could be spared from the divisor; we therefore had to obtain all but the last two figures of the quotient in the usual manner. In D, the divisor was made to consist of nine figures, to enable us to cut off at once. In fact, we had one figure (3) more than we actually used.

Compare the following: $46 \div \cdot 007\dot{5}\dot{1}$ to 3 places. The number of figures to be retained in the divisor depends not on the number of decimal places, but on the number of significant figures required in the quotient. It is therefore necessary to ascertain the *position* of the first of these significant figures. Place the decimal point in the divisor after its first significant figure, and shift it in the dividend the same number of places in the same direction, thus multiplying or dividing both by the same number. Thus we obtain, $46000 \div 7\cdot\dot{5}\dot{1}$. Beginning to divide, we find the first figure to be 6 thousands. Thus we shall require in the quotient four integral figures and the three decimal places; in all, six more figures. We further require the two initial figures of the divisor, which must consequently consist of $4 + 3 + 2 = 9$ figures; but then we begin to cut off at once.

```
7·5151515151) 46000·0000(6120·9677
               909 0909          8
               157 5757
                 7 2727
                   5091
                    582
                     56            Ans. 6120·968.
```

Wording: for first remainder—6, carry 1; 30, 31 and 9′ is 40, &c.; second remainder—5, carry 1; 2 and 7′ is 9, &c.; third remainder—2; 10 and 7′, &c.; and so on.

$\cdot 11 \div 1937{\cdot}\dot{4}3\dot{7}$ to 8 places. First place the point in the divisor after the 9, and move it two places to the left in the dividend also: $\cdot 0011 \div 19{\cdot}37\dot{4}3\dot{7}$.*

Wording: for first figure of quotient—19 in 0, ·0′; in 0, 0′; in 1, 0′; in 11, 0′; in 110, 5′;

∴ the quotient begins ·00005, and we therefore require four figures after the ciphers, and must retain 6 figures in the divisor.

```
19·3743) ·00110000 (·000056776
            13128           8
             1504
              148
               13            Ans. ·00005678.
```

Rule: In contracted division of decimals:

First, shift the decimal point in both divisor and dividend the same number of places and in the same direction, so as to bring it in the divisor into the most convenient place for ascertaining the denomination of the first significant figure of the quotient; we are then able to determine the number of significant figures required in the quotient.

Secondly, retain in the divisor, if long enough, two more places than this required number of significant figures. If not long enough, obtain the earlier figures of the quotient by "bringing down," and begin to "cut off" when the figures of the divisor are two more than those of the quotient yet to be found. If long enough, begin to "cut off" at once.

* Where the first significant figure of the divisor is 1, followed by a large digit, as here, it is better to take the first two figures as a trial divisor.

Thirdly, multiply each figure as it is "cut off" by the new figure of the quotient to carry the nearest ten.

·Exercise XXIII. (*a*).

(1) $862 \div 41\cdot 8\dot{1}7\dot{4}$ to 4 places.

(2) $437 \div 215\cdot 253$ to 3 places.

(3) $6 \div \cdot 1573$ to 3 places.

(4) $\cdot\dot{7}2\dot{6} \div \cdot\dot{0}47\dot{3}$ to 4 places.

(5) $\cdot 0041\dot{6} \div \cdot 08\dot{3}$ to 5 places.

(6) $1 \div \cdot\dot{1}23\dot{4}$ to 5 places.

(7) $54 \div \cdot 00037\dot{1}$ to the nearest unit.

(8) $\cdot 7\dot{2}8\dot{3} \div 4\cdot 562$ to 5 places.

(9) $\cdot\dot{4}6153\dot{8} \div \cdot\dot{5}3846\dot{1}$ to 6 places.

(10) $\cdot 0053 \div 72654$ to 8 places.

(11) $\cdot\dot{3} \div \cdot\dot{1}4285\dot{7}$ to 6 places.

By vulgar fractions and by decimals, correct to 5 places, find:

(12) $\frac{1}{3} \div \frac{1}{11}$; $\frac{4}{9} \div \frac{3}{7}$; $\frac{4}{9} \div \frac{2}{3}$; $7 \div \frac{1}{7}$; $\frac{1}{11} + \frac{1}{3}$; $\frac{3}{7} \div \frac{4}{9}$; $\frac{2}{3} \div \frac{4}{9}$; $\frac{1}{7} \div 7$ $\cdot 042 \div \frac{11}{300}$; $\frac{11}{300} \div \cdot 042$.

(13) If the length of the year be reckoned at $365\frac{1}{4}$ days, instead of its true length, $365\cdot 242264$ days, in what time will the error amount to 11 days, also to $2\cdot\dot{3}$ days?

Case IV. Division of a continued product.

Simplify to four places $\dfrac{\cdot 5\dot{8} \times \cdot\dot{0}1\dot{6} \times \cdot 2\dot{3}6\dot{8}}{\cdot\dot{0}1\dot{3}}$.

If we were to simplify the numerator to four places only, we should find our number of places insufficient, because some will disappear in shifting the decimal point to the right in both divisor and dividend; in other words, division by a proper fraction, which is equivalent to a multiplication by its reciprocal, an improper fraction, multiplies the error. Hence:

Rule: Choose any factor of the numerator as first multiplicand, then in it and in the divisor shift the decimal point so as to bring it in the divisor into the most convenient place for ascertaining the

denomination of the first significant figure of the quotient. This place is generally after the first or second significant figure of the divisor; thus:

$$\frac{\cdot 5\dot{8} \times \cdot\dot{0}1\dot{6} \times \cdot 2\dot{3}6\dot{8}}{\cdot\dot{0}1\dot{3}}$$

$$= \frac{236\cdot\dot{8}3\dot{6} \times \cdot 5\dot{8} \times \cdot\dot{0}1\dot{6}}{13\cdot\dot{0}1\dot{3}}$$

$$= \frac{23\cdot\dot{6}8\dot{3} \times \cdot 5\dot{8} \times \cdot\dot{1}6\dot{0}}{13\cdot\dot{0}1\dot{3}}$$

```
                23·6837
                61 061
                -------
                23 684
                14 210
                    24
                    14
                -------
                 3·7932
               98 885
               -------
               18 966
                3 034
                  303
                   30
                    3
              --------
     13·013)2·2336(·1716
               9323
                214
                 84
                  6
```

Ans. ·1716.

EXERCISE XXIII. (*b*).

Simplify:

(1) $\dfrac{\cdot 8\dot{1}7\dot{2} \times 10\cdot\dot{1}2\dot{3} \times \cdot 17\dot{8}\dot{4}}{\cdot 019\dot{2}}$ to 4 places.

(2) $\dfrac{\cdot 007\dot{4}\dot{2} \times \cdot 6\dot{7}0\dot{3} \times \cdot 0\dot{6}0\dot{1} \times \cdot 01}{1\cdot\dot{0}5\dot{7}}$ to 7 places.

(3) $\dfrac{7\cdot\dot{4}\dot{2} \times 6\cdot\dot{7}0\dot{3} \times 60\cdot\dot{1}6\dot{0} \times \cdot 01}{10570570\cdot\dot{5}7\dot{0}}$ to 7 places.

(4) $\dfrac{486\cdot 6875 \times 3\cdot 875 \times \frac{148}{385}}{100}$ to 4 places.

(5) $\dfrac{9328\cdot 7125 \times 3\cdot 48\dot{3} \times \frac{13}{73}}{100}$ to 4 places.

CHAPTER V.

PROGRESSIONS.

§ 1. ARITHMETICAL. It is required to add the following series:

2, 5, 8, 11, 14, 17, 20, 23. *Ans.* 100.

A series such as this, where each succeeding "term" is formed from the preceding one by adding or subtracting the same quantity, is called an ARITHMETICAL PROGRESSION, and the quantity invariably added or subtracted is called the Common Difference of the terms.

If no longer or more complicated series were ever proposed than that given above, there would be no need to generalize on the subject; but let it be proposed to add, 1, $3\frac{5}{8}$, $6\frac{1}{4}$, $8\frac{7}{8}$, &c., to 1000 terms. It would be laborious to write out the whole series in order to perform the addition. Examine the first series given; under it, write the same series in reverse order, and add the terms two and two; thus,

$$\begin{array}{cccccccc} 2, & 5, & 8, & 11, & 14, & 17, & 20, & 23 \\ 23, & 20, & 17, & 14, & 11, & 8, & 5, & 2 \\ \hline 25, & 25, & 25, & 25, & 25, & 25, & 25, & 25 \end{array} = 8 \times 25 = 200.$$

Hence the sum of the *two* series is 200; that of the one series is 100. The number 25 is obtained by adding the first and the last terms, or the second and the last but one, and so on; this 25 is multiplied by the number of terms, 8; and the product is divided by 2. The question now arises: Will this method hold for all arithmetical progressions?

The nature of an A.P. is such that the second term exceeds the first by the common difference, and the last but one falls short of the last by the same quantity; hence the sum of the first and last must be equal to that of the second and last but one, which again equals the sum of the third and last but two, since the third exceeds the second by as much as the last but two falls short of the last but one, and so on. Hence the sum of the double series is the sum of the first and last terms multiplied by the number of terms. The sum of the single series, then, will be found by multiplying the sum of the first and last terms by *half* the number of terms.

This conclusion is pictured by the following "formula:"

$$s = (a + l) \times \frac{n}{2},$$

where s stands for the sum of the series,
a „ first term,
l „ last term,
n „ number of terms.

By this formula we can sum a series where l, the last term, is known. But in the series, 1, $3\frac{5}{8}$, $6\frac{1}{4}$, &c., given above, l has yet to be found.

Let d represent the common difference; the first term is a; the second, $a + d$; the third, $a + 2 \times d$, &c. In other words, the series might be written thus:

1st term,	2nd term,	3rd term,	4th term,	5th term, &c.
a,	$a + d$,	$a + 2 \times d$,	$a + 3 \times d$,	$a + 4 \times d$, &c.,

where the number of d's added to a in each term is one less than the number of the term; thus the 20th term will be $a + 19 \times d$, and the 1000th term of the above series is $1 + 999 \times 2\frac{5}{8} = 2623\frac{3}{8}$. The sum of the series is then,

$$(1 + 2623\tfrac{3}{8}) \times \tfrac{1000}{2} = 2624\tfrac{3}{8} \times 500 = 1312187\tfrac{1}{2}.$$

The formula for the last, or nth term is, $l = a + (n - 1) \times d$.

Exercise XXIV.

(1) Find the sum of 1, 2, 3, &c., to 1000 terms.

(2) Find the seventieth odd number.

(3) Find the sum of 100 terms of 1, 3, 5, &c.

(4) „ 60 „ $3\frac{1}{2}$, $4\frac{3}{4}$, 6, &c.

(5) „ 75 „ ·16, ·18, ·2, &c.

(6) Find the eightieth term of 2·5, 2·75, 3, &c.

(7) If I invest in a Building Society 10*s.* a month for the 1st year, £1 a month for the 2nd year, £1. 10*s.* a month for the 3rd year, and so on, what will be my payment in the 10th year, and how much shall I have invested altogether at the end of the 10th year?

(8) If a stone fall through 16·1 ft. in the 1st second of time, 48·3 ft. in the 2nd second, 80·5 ft. in the 3rd second, and so on, how deep will be the shaft of a mine where a stone takes 7 seconds to reach the bottom?

9) Prove that in a descending series $l = a - (n-1) \times d$, and, as before, $s = (a + l) \times \frac{n}{2}$.

§ 2. GEOMETRICAL SERIES ASCENDING. A series where each new term is formed from the preceding term by multiplication instead of addition is called a GEOMETRICAL PROGRESSION, and the number by which we multiply each time is called the "common ratio." Thus 5, 50, 500, &c., is a geometrical progression whose common ratio is 10.

The multiples of a number are in arithmetical, the powers in geometrical progression; e.g.,

0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30......81, &c., are in A.P.
1, 3, 9, 27, ...81, &c., are in G.P.

It is required to sum the series, 4, 12, 36, 108, &c., to 6 terms. The common ratio in this series is 3; therefore the series may be written:

1st term,	2nd term,	3rd term,	4th term,	5th term,	6th term,
4	4×3	4×3^2	4×3^3	4×3^4	4×3^5

whence we see that the power of the common ratio in each term is one less than the number of the term. Thus the 20th term would be 4×3^{19}; or if l be the nth term, and r the common ratio, $l = a \times r^{n-1}$.

Let s be the sum of the six terms, then

$$s = 4 + 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + 4 \times 3^4 + 4 \times 3^5$$

If this series is multiplied by the common ratio 3, we obtain,

$$3 \times s = 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + 4 \times 3^4 + 4 \times 3^5 + 4 \times 3^6$$

Subtracting s from $3 \times s$, we get $2 \times s$, and subtracting from

$$\begin{array}{r} 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + 4 \times 3^4 + 4 \times 3^5 + 4 \times 3^6 \\ \underline{4 + 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + 4 \times 3^4 + 4 \times 3^5 \qquad\quad} \end{array}$$

we find that the intermediate terms cancel one another, and we have to subtract 4 from 4×3^6. Hence,

$$2 \times s = 4 \times 3^6 - 4, \text{ and } s = \frac{4 \times 3^6 - 4}{2} = 1456. \qquad \textit{Ans. } 1456.$$

This conclusion is pictured by the following formula: $s = \frac{a \times r^n - a}{r - 1}$

where s stands for the sum of the series,
a „ first term,
r „ common ratio,
n „ number of terms.

Exercise XXV.

Sum the series:

(1) 1, 3, 9, &c., to 8 terms.

(2) ·001, ·01, ·1, &c., to 10 terms.

(3) 5, 5^2, 5^3, &c., to 5 terms.

(4) 1, 1·05, 1·05 × 1·05, &c., to 6 terms, correct to 4 places.

§ 3. Geometrical Series descending. In a G. P. where the common ratio is an aliquot* fraction, say $\frac{1}{r}$, each term will be less than the preceding term, and the G.P. may be considered as one formed by division by r, instead of multiplication by $\frac{1}{r}$. Consider the series, $s = 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$.

Here each term is formed by dividing the preceding term by 3. If the series be multiplied by 3 (the common divisor), each term will become the one preceding it, and we obtain, calling s the sum of the above series,

$3 \times s = 27 + 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}$. Subtracting,
$1 \times s = \quad 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$, we obtain,
$(3 - 1) \times s = 2 \times s = 27 - \frac{1}{81} = 26\frac{80}{81}$, and $\therefore s = 13\frac{40}{81}$.

Exercise XXVI.

Sum the series:

(1) 1, $\frac{1}{3}$, $\frac{1}{9}$, &c., to 8 terms.

(2) 1, $\frac{1}{5}$, $\frac{1}{5^2}$, &c., to 5 terms.

(3) 1000, 100, 10, &c., to 10 terms.

(4) 3, ·3, ·03, &c., to 10 terms.

* An aliquot fraction is a fraction which is a measure of unity, and its reciprocal is consequently an integer.

§ 4. ENDLESS GEOMETRICAL SERIES.

Examine the series, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$ &c.

$$\begin{array}{ll}
1=1 & =2-1\\
1+\frac{1}{2}=1\frac{1}{2} & =2-\frac{1}{2}\\
1+\frac{1}{2}+\frac{1}{4}=1\frac{3}{4} & =2-\frac{1}{4}\\
1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=1\frac{7}{8} & =2-\frac{1}{8}\\
1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=1\frac{15}{16} & =2-\frac{1}{16}\\
1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=1\frac{31}{32} & =2-\frac{1}{32} \text{ and so on.}
\end{array}$$

Let us apply the method of § 3 to the series, $1+\frac{1}{2}+\frac{1}{4}+$ &c. We must stop somewhere. Let us stop at $\frac{1}{32}$.

$$\begin{aligned}
s &= \phantom{2+{}} 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}\\
2\times s &= 2+1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}\\
\hline
s &= 2-\tfrac{1}{32}, \text{ as above.}
\end{aligned}$$

Each successive series, then, falls short of 2 by its last term; but this last term continually diminishes, and can be made smaller than any assigned quantity. The series is therefore an approximation whose LIMIT is 2 (p. 119).

Again, take the series,

$$\begin{aligned}
s &= \phantom{27+{}} 9+3+1+\tfrac{1}{3}+\tfrac{1}{9}+\tfrac{1}{27}+\tfrac{1}{81}\,\Big|+\text{&c.}\\
3\times s &= 27+9+3+1+\ldots\ldots\ldots\tfrac{1}{27}\Big|+\text{&c.}\\
\hline
(3-1)\times s &= 2\times s = 27-\tfrac{1}{81}
\end{aligned}$$

We see that $2\times s$ falls short of 27 by the last term, and hence, since again the last term may be made less than any assigned quantity, the limit of $2\times s$ is 27, and of s it is $13\frac{1}{2}$.

Find the limit of $7+1\frac{2}{5}+\frac{7}{25}+$ &c. Here $\frac{1}{r}=\frac{1}{5}$ or $r=5$.

$$\begin{aligned}
5\times s &= 35+7+1\tfrac{2}{5}+\ldots\ldots\\
1\times s &= \phantom{35+{}} 7+1\tfrac{2}{5}+\ldots\ldots\\
\hline
4\times s &= 35-(\text{some quantity which may be made as}
\end{aligned}$$

small as we please). Hence 35 is the limit of $4\times s$, and the limit of $s=8\frac{3}{4}$.

EXERCISE XXVII.

Find the limits of:

(1) $1+\frac{1}{5}+\frac{1}{5^2}+$ &c.

(2) $3\frac{1}{2}+\frac{1}{2}+\frac{1}{14}+$ &c.

(3) $12+3+\frac{3}{4}+$ &c.

(4) $64+8+1+$ &c.

(5) $\cdot 5+\cdot 25+\cdot 125+$ &c.

(6) $20+6\frac{2}{3}+2\frac{2}{9}+$ &c.

§ 5. Limit of Recurring Decimals.

Every recurring decimal is a G.P. where r is a power of 10; thus, $\cdot\dot{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$, r being 10; $\cdot\dot{4}6\dot{8} = \frac{468}{1000} + \frac{468}{1000000} + \ldots$, r being 1000.

Case I. Find the limit of $\cdot\dot{3}$.

$$\begin{aligned} s &= \cdot 333\ldots\ldots \\ 10 \times s &= 3\cdot 333\ldots\ldots \\ \hline 9 \times s &= 3 \qquad s = \tfrac{3}{9} = \tfrac{1}{3}. \end{aligned}$$

Ans. $\frac{1}{3}$.

Find the limit of $\cdot\dot{4}6\dot{8}$.

$$\begin{aligned} s &= \cdot 468468468\ldots\ldots \\ 1000 \times s &= 468\cdot 468468468\ldots\ldots \\ \hline 999 \times s &= 468 \qquad s = \tfrac{468}{999} \Big| \tfrac{52}{111}. \end{aligned}$$

Ans. $\frac{52}{111}$.

Exercise XXVIII.

Find the limits of the following:

(1) $\cdot\dot{3}$.	(7) $\cdot\dot{2}\dot{7}$.	(13) $\cdot\dot{1}4285\dot{7}$.	(19) $\cdot\dot{0}136986\dot{3}$.
(2) $\cdot\dot{7}$.	(8) $\cdot\dot{7}\dot{2}$.	(14) $\cdot\dot{8}5714\dot{2}$.	(20) $\cdot\dot{0}7936\dot{5}$.
(3) $\cdot\dot{1}$.	(9) $\cdot\dot{1}3\dot{5}$.	(15) $\cdot\dot{4}2857\dot{1}$.	(21) $\cdot\dot{0}028\dot{7}$.
(4) $\cdot\dot{2}$.	(10) $\cdot\dot{2}3\dot{4}$.	(16) $\cdot\dot{1}5384\dot{6}$.	(22) $\cdot\dot{0}1098\dot{9}$.
(5) $\cdot\dot{6}$.	(11) $\cdot\dot{0}2\dot{4}$.	(17) $\cdot\dot{1}44\dot{1}$.	(23) $\cdot\dot{0}421\dot{2}$.
(6) $\cdot\dot{9}$.	(12) $\cdot\dot{0}7\dot{4}$.	(18) $\cdot\dot{0}243\dot{9}$.	(24) $\cdot\dot{0}0025\dot{9}$.

It will be observed that the result uniformly must be this: the figures of the recurring period give the numerator, and the denominator consists of as many nines as there are recurring figures.

Case II. Find the limit of $\cdot 000\dot{2}\dot{7}$.

$$\begin{aligned} s &= \cdot 000272727\ldots\ldots \\ 100 \times s &= \cdot 027272727\ldots\ldots \\ \hline 99 \times s &= \cdot 027 \qquad s = \tfrac{\cdot 027}{99} = \tfrac{27}{99000} \Big| \tfrac{3}{11000}. \end{aligned}$$

Exercise XXIX.

Find the limits of:

(1) $\cdot 0\dot{3}$.	(3) $\cdot 00\dot{3}\dot{6}$.	(5) $\cdot 00\dot{1}221\dot{3}$.	(7) $\cdot 00\dot{1}0\dot{8}$.
(2) $\cdot 07\dot{2}$.	(4) $\cdot 0\dot{0}3\dot{6}$.	(6) $\cdot 0000\dot{9}$.	(8) $\cdot 000\dot{0}10\dot{8}$.

It will be observed that the numerator consists of the recurring figures, and the denominator of as many nines as there are recurring figures, followed by the number of non-recurring ciphers in the decimal.

CASE III. Find the limits of $\cdot 48\dot{3}2\dot{4}$.

$$s = \cdot 48324324324\ldots\ldots$$
$$1000 \times s = 483\cdot 24324324324\ldots\ldots$$
$$999 \times s = 482\cdot 76$$
$$s = \frac{482\cdot 76}{999} = \frac{48276}{99900} \Big| \frac{5364}{11100} \Big| \frac{447}{925}. \qquad \textit{Ans. } \frac{447}{925}.$$

EXERCISE XXX.

Find the limits of the following:

(1) $\cdot 1\dot{3}\dot{6}$.	(5) $\cdot 04\dot{7}\dot{2}$.	(9) $\cdot 225\dot{9}$.	(13) $\cdot 91\dot{6}$.
(2) $\cdot 6\dot{2}\dot{7}$.	(6) $\cdot 06\dot{5}6\dot{3}$.	(10) $\cdot 58\dot{3}$.	(14) $\cdot 08\dot{3}$.
(3) $\cdot 4\dot{7}\dot{2}$.	(7) $\cdot 22\dot{5}\dot{9}$.	(11) $\cdot 41\dot{6}$.	(15) $\cdot 8\dot{3}$.
(4) $\cdot 47\dot{2}$.	(8) $\cdot 2\dot{2}5\dot{9}$.	(12) $\cdot 001\dot{6}$.	(16) $\cdot 5\dot{8}\dot{3}$.

§ 6. The question we have been solving in § 5 might have been stated thus: What vulgar fraction would have produced the given recurring decimal? And the result might have been obtained experimentally, thus:

$$\frac{1}{9} = \cdot\dot{1} \quad \therefore \quad \frac{2}{9}, \ \frac{3}{9}, \ \frac{4}{9}, \quad \&c. = \cdot\dot{2}, \ \cdot\dot{3}, \ \cdot\dot{4}, \ \&c.$$
$$\frac{1}{99} = \cdot\dot{0}\dot{1} \quad \therefore \quad \frac{2}{99}, \ \frac{13}{99}, \ \frac{75}{99}, \quad \&c. = \cdot\dot{0}\dot{2}, \ \cdot\dot{1}\dot{3}, \ \cdot\dot{7}\dot{5}, \ \&c.$$
$$\vdots$$
$$\frac{1}{99999} = \cdot\dot{0}000\dot{1} \quad \therefore \quad \frac{2}{99999}, \ \frac{7628}{99999}, \ \&c. = \cdot\dot{0}000\dot{2}, \ \cdot\dot{0}762\dot{8}, \ \&c.$$

In fact, in dividing 1000 by 999 we obtain quotient 1 and remainder 1, which, being again multiplied by 1000, yields the same quotient with the same remainder, and so on. Again, in dividing 43000 by 999, we obtain quotient 43, and remainder 43; hence $\frac{43}{999} = \cdot\dot{0}4\dot{3}$. Similarly, $\frac{285}{999} = \cdot\dot{2}8\dot{5}$; $\frac{4723}{9999} = \cdot\dot{4}72\dot{3}$, and so on; which leads to the conclusion stated at the end of Case I.

For Case II., since $\cdot 00\dot{7}5\dot{1} = \cdot\dot{7}5\dot{1} \div 100$, and $\cdot\dot{7}5\dot{1} = \frac{751}{999}$, $\therefore\ \cdot 00\dot{7}5\dot{1} = \frac{751}{999} \div 100 = \frac{751}{99900}$.

For Case III., $\cdot 84\dot{3}5\dot{1} = \cdot 84 + \cdot 00\dot{3}5\dot{1} = \frac{84}{100} + \frac{351}{99900}$

$$= \frac{999 \times 84}{99900} + \frac{351}{99900} = \frac{84 \times 1000 - 84 + 351}{99900} = \frac{84351 - 84}{99900} = \frac{84267}{99900}.$$

This leads to the following general rule: For the numerator, subtract the non-recurring figures from the decimal, as given to the end of the first period. For the denominator, write as many nines as there are recurring figures, followed by as many ciphers as there are non-recurring figures.

CHAPTER VI.

PROPERTIES OF DECIMALS.

§ 1. We must premise that the vulgar fractions on which we shall reason in this Chapter are all at lowest terms, unless the contrary is stated.

§ 2. In Chaps. II. and III., all the decimal fractions were terminating; the subsequent Chapter treated of non-terminating decimals. The question arises: Can we, by mere inspection of a given vulgar fraction, determine the nature of the resulting decimal?

Decimalize $\frac{5}{8}, \frac{13}{25}, \frac{3}{7}, \frac{19}{28}$.

UNITS. TENTHS. HUNDREDTHS. THOUSANDTHS.

$$a.\quad \frac{5}{8} = \frac{5 \times \overset{5}{\cancel{10}}}{\underset{4}{\cancel{8}}} = \frac{5 \times 5 \times \overset{5}{\cancel{10}}}{\underset{2}{\cancel{4}}} = \frac{5 \times 5 \times 5 \times \overset{5}{\cancel{10}}}{\underset{1}{\cancel{2}}} = \cdot 625.$$

$$b.\quad \frac{13}{25} = \frac{13 \times \overset{2}{\cancel{10}}}{\underset{5}{\cancel{25}}} = \frac{13 \times 2 \times \overset{2}{\cancel{10}}}{\underset{1}{\cancel{5}}} = \quad \cdot\ \cdot\ \cdot\ \cdot\quad \cdot 52.$$

$$c.\quad \frac{3}{7} = \frac{3 \times 10}{7} = \frac{3 \times 10 \times 10}{7} = \frac{3 \times 10 \times 10 \times 10}{7} = \cdot 428\ldots$$

$$d.\quad \frac{19}{28} = \frac{19 \times \overset{5}{\cancel{10}}}{\underset{14}{\cancel{28}}} = \frac{19 \times 5 \times \overset{5}{\cancel{10}}}{\underset{7}{\cancel{14}}} = \frac{19 \times 5 \times 5 \times 10}{7} = \cdot 678\ldots$$

Fractions a and b give terminating decimals, because each successive introduction of 10 into the numerator gets rid of one 2 or one 5 in the denominator. If, then, the denominator has no other prime factors than 2 or 5, the decimal will terminate.

In fraction c, the denominator is prime to 10; hence no introduction of 10 into the numerator will cancel or reduce it.

In fraction d, the factors of the denominator are $2 \times 2 \times 7$, of which the 2×2 are cancelled by the two successive introductions of 10, but the 7 remains unaffected. From this we see that the solution of the question proposed depends solely upon the denominator.

Learn by heart: *If the denominator of a fraction contain no prime factors other than* 2 *or* 5, *its decimal will terminate; and if it contain others, it will not terminate.*

§ 3. Every introduction of 10 into the numerator will cancel 2, 5, or 2×5 in the denominator, and will yield one decimal place; hence there will be a decimal place for every 2×5 or 0 in the denominator, and a decimal place for every 2 or 5 after the ciphers are allowed for. (N.B. After accounting for the ciphers there cannot be twos *and* fives.)

§ 4. Notice that if the denominator, after disregarding any ciphers at the end, is a power of 2, the last figure of the decimal must be 5; if a power of 5, it must be even.

§ 5. We have now to consider the non-terminating fractions, and the question arises whether the figures will necessarily recur, or whether they will follow some other kind of arrangement, or no arrangement at all.

If in division by any number, say 7, 13, &c., any remainder occur a second time, the figures in the quotient, i.e. in the decimal, must thenceforward be the same as those following the previous occurrence of that remainder. Take, for example, $\frac{3}{7}$ and $\frac{39}{41}$.

```
7)30(·428571, &c.        41)390(·95121, &c.
   20                        210
    60                         50
     40                         90
      50                         80
       10                         39 &c.
        3 &c.               Ans. ·95121.
  Ans. ·428571.
```

(Recurring dots over the first and last figures: $\dot{4}2857\dot{1}$ and $\dot{9}512\dot{1}$.)

In dividing by 7, there cannot be more than *six* different remainders, viz. 1, 2, 3, 4, 5, 6; in dividing by 41, there cannot be more than forty, viz. 1, 2, 3,......40. If, then, we should have performed respectively six or forty steps with different remainders, the next remainder must be one that has already occurred. Hence $\frac{3}{7}$ and $\frac{39}{41}$ must yield recurring decimals, and the recurrence must take place not later than after the sixth and fortieth places respectively. We see, however, from $\frac{39}{41}$, that it may take place earlier. Thus,

With denominator 7, recurrence will take place at latest after 6 places.
„ 41, „ „ 40 „
and generally, n, „ „ $(n-1)$ „

§ 6. If we take a number prime to 10, say 7, the L. C. M. of 10 and 7 is 70 (Part I. Ch. XI. § 18), and all tens below 70, viz. 10, 20, 30, 40, 50, 60, must give different remainders when divided by 7; for if any two gave the same remainder, their difference, a number of tens less than 70, would be a multiple of 7 (Part I. Ch. XI. 2nd part of § 7), which is impossible. These remainders, then, must be, in some order or other, the numbers 1, 2, 3, 4, 5, 6. Thus:

10	gives remainder	3,	40	gives remainder	5,
20	„	6,	50	„	1,
30	„	2,	60	„	4.

This means that each multiple of ten is a number of sevens + the corresponding number of units placed beside it. If, now, any two numbers of the first column, say 10 and 20, be multiplied together, we have to multiply (a number of sevens + 3) by (a number of sevens + 6); the multiplication by sevens gives sevens; the multiplication by 6 gives a number of sevens + 6 × 3. Similarly, the product of all these multiples of ten will be a number of sevens + the product of all the units to the right, or $10 \times 20 \times 30 \times 40 \times 50 \times 60 =$ a number of sevens $+ 1 \times 2 \times 3 \times 4 \times 5 \times 6$. Hence, $10 \times 20 \times 30 \times 40 \times 50 \times 60 - 1 \times 2 \times 3 \times 4 \times 5 \times 6$ is a multiple of 7. But $10 \times 20 \times 30 \times 40 \times 50 \times 60 = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 1000000$, and $\therefore 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 1000000 - 1 \times 2 \times 3 \times 4 \times 5 \times 6$, or $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 999999$, must be a multiple of 7. Seven, then, cannot be prime both to $1 \times 2 \times 3 \times 4 \times 5 \times 6$ and to 999999. (Part I. p. 147, note 2.)

Suppose the number we have chosen, as in this case 7, be not only prime to 10, but a prime number, it is prime to the first of these quantities, being prime to each of its factors; it must therefore divide the second, viz. 999999, which is formed by writing six nines in succession. Similar reasoning will shew that 13 will divide 999999999999 (twelve nines); 17 will divide sixteen nines; and, generally, if n is prime, n will divide $(n-1)$ nines in succession. (Fermat's Theorem.)

§ 7. Since 7 is a measure of 999999 (six nines), 13 of 999999999999 (twelve nines), and n* of $(n-1)$ nines, $\therefore$ 1000000 $(10^6) \div 7$ leaves remainder 1, $10^{12} \div 13$ leaves remainder 1, and $10^{n-1} \div n$* leaves remainder 1.

$$\begin{array}{lll} \text{Again, } 3000000 \; (= 3 \times 10^6) \div 7 & \text{leaves remainder} & 3, \\ 5 \times 10^{12} \div 13 & \quad " & 5, \\ \dagger a \times 10^{n-1} \div n^* & \quad " & a. \end{array}$$

Hence, in dividing by 7, 13, n, after 6, 12, $n-1$ places, we shall have as remainder the original dividend, and the quotient, i.e. the decimal fraction, will recur from the beginning, and have 6, 12, $n-1$ figures in the "period."

A prime number n, then (other than 2 or 5), will yield a pure circulator, having $n-1$ recurring figures. Thus $\frac{6}{7} = \cdot\dot{8}5714\dot{2}$ (6 figures), $\frac{5}{13} = \cdot\dot{3}8461538461\dot{5}$ (12 figures). This last decimal, however, can be indicated thus, $\cdot\dot{3}8461\dot{5}$ (6 figures); and generally, though n must yield a decimal *expressible* with $n-1$ recurring figures, it may be possible to use fewer figures in the period; but in this case this smaller number must consequently be a measure of $n-1$; or, in other words, though n must measure $n-1$ nines, it may measure an aliquot fraction of that number of nines.

§ 8. If $\frac{1}{7}$ be decimalized, we obtain $\cdot\dot{1}4285\dot{7}$, a period of six figures; hence all the remainders must have occurred. If, then, we wish to decimalize $\frac{2}{7}$, $\frac{3}{7}$, &c., we shall only have the same succession of figures with different commencements. Writing the period thus,

$$\begin{array}{ccc} 7 & 1 & 4 \\ 5 & 8 & 2 \end{array}$$

we can from it read off the decimal for any number of sevenths.

* n being *prime*. † a being less than n.

Thus $\frac{2}{7}$ will begin with 2, $\frac{3}{7}$ with 4, $\frac{4}{7}$ with 5, &c., and this first figure can at once be ascertained by commencing the division. This ring should be learnt by heart.

Now decimalize $\frac{1}{13}$. $\frac{1}{13} = \cdot\dot{0}7692\dot{3}$.

$$\begin{array}{l} 13)\,100\,(\cdot\dot{0}7692\dot{3} \\ \quad\quad 90 \\ \quad\quad 120 \\ \quad\quad\quad 30 \\ \quad\quad\quad\ 40 \\ \quad\quad\quad\quad 1, \text{\&c.} \end{array}$$

This period contains six figures only, therefore only six of the 12 possible remainders can have occurred; and if we try another number of 13ths, we cannot be sure that that numerator will be one of these remainders. Take $\frac{2}{13}$.

$$\begin{array}{l} 13)\,20\,(\cdot\dot{1}5384\dot{6} \\ \quad\quad 70 \\ \quad\quad 50 \\ \quad\quad 110 \\ \quad\quad\quad 60 \\ \quad\quad\quad\ 80 \\ \quad\quad\quad\quad 2, \text{\&c.} \end{array}$$

Here we find the other six remainders; and now no fraction whose denominator is 13 can be proposed whose numerator is not in one of the two sets of remainders. From what was said in § 5, it follows that the two series can at no point coincide. For 13, we shall then require these two rings:

$$\begin{array}{ccc} 3 & 0 & 7 \\ 2 & 9 & 6 \end{array} \qquad\qquad \begin{array}{ccc} 6 & 1 & 5 \\ 4 & 8 & 3 \end{array}$$

As these figures are not all different, in two of the cases the first *two* figures must be found by actual division. If there occur in any set of rings similar sequences of *two* figures each, the first *three* figures must be found by actual division, and so on.

§ 9. 365 days $= 5 \times 73$ days. Fractions with denominator 73 occur, therefore, frequently in the calculation of interest. 73 has eight recurring figures, and will therefore require $\frac{72}{8} = 9$ rings to exhaust the 72 remainders.

3	0	1	7	2	6	4	5	2	3	1	5	1	7	8
6	A	3	9	B	0	9	C	0	9	D	0	9	E	0
8	9	6	3	7	2	7	4	5	4	8	6	1	2	8

4	1	0	7	1	2	6	1	6	4	2	4
0	F	9	6	G	3	5	H	4	3	K	6
9	8	5	7	8	2	3	8	3	5	7	5

From these rings, any fraction with denominator 73 can be read off as a decimal, after the first two figures have been found by actual division.

§ 10. Examination of all these rings shews the peculiar property that the opposite numbers added together make 9. Thus the ring for 7 consists of the two halves 142 and 857, whose sum is 999. The question arises: Is this a mere accident or a property belonging to other than these fractions; and if so, to what others?

In Ch. V., we have found that every pure circulator can be expressed as a vulgar fraction having for denominator as many nines as there were recurring figures, and for numerator the recurring period. Thus $\frac{2}{7}$ or $\cdot\dot{2}8571\dot{4} = \frac{285714}{999999}$. This fraction is of course reducible to lower terms, where the new denominator is a measure of 999999. Now $999999 = 999 \times 1001$; seven being prime, it measures either 999 or 1001. We have already found that 7 gives six decimal places, i. e. that 999999 is the *least* number of nines divisible by 7; hence 7 does not measure 999, and must measure 1001. (Part I. p. 147, note 2). Since $\frac{285714}{999999} = \frac{285714}{999 \times 1001}$ will reduce to $\frac{2}{7}$, the factor 999 must be entirely cancelled out; hence the numerator must be divisible by 999. Let us, then, investigate the criterion for divisibility by 999.

$1000 \div 999$ leaves remainder 1
$7000 \div 999$ „ 7

$123325551 = 123000000 + 325000 + 551.$

$123000000 \div 999$	leaves remainder	123
$325000 \div 999$	„	325
$551 \div 999$	„	551
	Sum of the remainders,	999

Similarly 863584284267 ÷ 999 gives remainder 863 + 584 + 284 + 267 = 1998, and as this is divisible by 999, the number is so.

Hence a number is divisible by 999 (three nines) if the sum of its digits, added in sets of three figures beginning at the units' place, is divisible by 999.

Similar reasoning will shew that a number is divisible by 99, 9999, 99999, &c., if the sum of the digits, added up in sets of 2, 4, 5 figures respectively, be divisible by 99, 9999, 99999, &c. (Cf. the criterion for divisibility by 9 given in Part I. p. 132.)

Now we have shewn that 285714 must be divisible by 999, and therefore its digits added up in sets of three figures must be divisible by 999; and as we have but six figures, the sum of the two sets cannot exceed 999, and must therefore be 999. This can only take place if the opposite figures, as placed in the ring, make up together the number 9. (We shall call two figures whose sum is 9 complemental to 9.)

The conditions under which this process of reasoning holds good are, (*a*) that the number of recurring figures of the period is *even*, because then its corresponding nines can be broken up into two factors of which one is half the number of nines; (*b*) that the denominator of the vulgar fraction from which the recurring decimal is derived is prime to that less number of nines.

If, then, we know of any prime number that it will yield an *even* number of recurring figures, the halves of the period will be complemental, and accordingly we shall only need to work out by division the first half of the period. For example, of $\frac{19}{73}$, only four places, viz. 2602, need be found by division,

$$\begin{array}{l} 73)190(\cdot\dot{2}602|739\dot{7} \\ \quad\;\; 440 \\ \quad\quad 200 \\ \quad\quad\; 54 \end{array}$$

the other figures, 7397, being complemental. 11 has two recurring figures which must therefore be complemental, and only one needs to be found by division.

§ 11. The following table shews the number of recurring figures belonging to the first twenty-five primes, omitting 2 and 5.

PRIMES AND THEIR DECIMAL PLACES.

3...... 1	19...... 18	41...... 5	61......60	83......41
7...... 6	23......22	43......21	67......33	89......44
11...... 2	29......28	47......46	71......35	97......96
13...... 6	31......15	53......13	73...... 8	101...... 4
17......16	37...... 3	59......58	79......13	103......34

From this table we can extend the criteria for divisibility by prime numbers (Part I. Ch. XI. § 8); e.g., a number is divisible by 37 if the sum of its digits, added in sets of three beginning at the units' place, be divisible by 37; for 37 has three recurring figures, i.e. $1000 \div 37$ gives remainder 1, &c.

§ 12. Decimalize $\frac{63}{137}$.

$$
\begin{array}{l}
137)630(\cdot\dot{4}598540\dot{1} \\
\quad\;\; 820 \\
\quad\;\; 1350 \\
\quad\;\;\; 1170 \\
\quad\;\;\;\; 740 \\
\quad\;\;\;\;\; 550 \\
\quad\;\;\;\;\;\; 200 \\
\quad\;\;\;\;\;\;\; 63
\end{array}
$$

The successive dividends are,

$$
\begin{array}{rrrr}
63, & 82, & 135, & 117 \\
74, & 55, & 2, & 20 \\
\hline
137, & 137, & 137, & 137
\end{array}
$$

Notice that the eight dividends or remainders, corresponding to the eight recurring complementals, added in pairs as above, make up the divisor 137. This depends on the following considerations:

a. 99999999 is divisible by 137.

b. $99999999 = 9999 \times 10001$.

c. 137 does not measure 9999, for otherwise it would have only 4 recurring figures in its period; therefore, being prime, it must measure 10001.

d. $10001 \div 137$ leaves no remainder, $\therefore$ $10000 \div 137$ will leave a remainder 136, being short by 1 of the above multiple of 137.

e. $63 \times 10001 \div 137$ leaves no remainder, $\therefore$ $63 \times 10000 \div 137$ leaves remainder $137 - 63$, being 63 short of the multiple that 63×10001, or 630063, is of 137.

This means that the remainder after four steps must fall short of 137, the divisor, by 63, the first dividend. Had we begun with 82,

the second dividend, four steps would have led us to the same conclusion, ∴ after any four steps the dividend at the commencement added to the last remainder must give the divisor.

As this property depends on the same conditions as that investigated in § 10, it is subject to the same limitations. Hence, wherever the recurring figures are complemental to 9, the corresponding remainders will be complemental to the divisor.

§ 13. Reduce $\frac{11}{21}$ to a decimal.

$$\begin{array}{l} 21)110(\cdot\dot{5}2380\dot{9} \\ \quad\ \ 50 \\ \quad\ \ \ 80 \\ \quad\ \ \ 170 \\ \quad\ \ \ \ \ 200 \\ \quad\ \ \ \ \ \ 110 \end{array}$$

Observe that this fraction, although 21 is not prime, yields a pure circulator. We might have judged, à priori, that such would be the case, for 21 is prime to 10, ∴ 210 is L.C.M. of 21 and 10, and no two multiples of 10 less than 210 can give the same remainder (§ 6); consequently any remainder, say 2, can only have been derived from the remainder 17, 17 only from 8, 8 only from 5, and 5 only from 11; hence the recurrence which must take place must be referable to the first dividend 11.

The statement in § 7, that a prime yields a pure circulator may now be extended to numbers prime to 10.

Again, observe that the recurring figures are not complemental to 9, nor the dividends to 21, 21 not being prime to 999.

§ 14. If the denominator contain as factors twos or fives with other primes, the resulting decimal will be a mixed circulator; for, dividing first by the twos or fives (§ 2), we shall obtain a terminating decimal; and if this is then divided by the remaining factor, which is prime to 10, we get a pure circulator commencing at the end of the terminating decimal previously obtained; e.g., $\frac{37}{52} = \frac{37}{4 \times 13} = \frac{9 \cdot 25}{13} = \cdot 71\dot{1}5384\dot{6}$.

$$\begin{array}{l} 4)\underline{37\quad} \\ 13)\underline{9 \cdot 25\qquad\quad} \\ \qquad \cdot 71\dot{1}5384\dot{6} \end{array}$$

two non-recurring figures due to 4 (§ 3), followed by six recurring *figures due to 13* (see table, § 11).

§ 15. It is now required to determine the number of recurring figures in the period derived from a vulgar fraction whose denominator, though composite, is prime to 10, or, which comes to the same thing, the least number of nines divisible by this denominator.

Decimalize $\frac{1}{7 \times 73} = \frac{1}{511}$. We know that 6 and 8 nines are the least number of nines divisible by 7 and 73 respectively. Hence 24 nines (24 being L. C. M. of 6 and 8) is the least number of nines divisible by 7 and 73, and $\therefore$ by 511,* or 511 will have 24 recurring figures.

Generally: A denominator which is a product of *different* primes other than 2 and 5 will have in the period as many recurring figures as is equal to the L. C. M. of the numbers of recurring figures proper to each prime. These conclusions do not apply to denominators containing as factors powers of prime numbers, the condition in the foot-note not being fulfilled.

[If p be a prime number with a period of k figures, then it can be proved that

p^2 will have a period of either ....k or $p \times k$ figures,
p^3 " " ...... k or $p \times k$ or $p^2 \times k$ "
p^4 " " k or $p \times k$ or $p^2 \times k$ or $p^3 \times k$ "
&c. &c.]

§ 16. Addition and subtraction of recurring decimals, where the whole period of the result is required, need only be carried out to the number of places represented by the L. C. M. of the numbers of recurring figures in each of the given quantities + the maximum number of non-recurring decimals. An example will render this obvious.

Add $\cdot 58\dot{3}$, $\cdot 0\dot{0}41\dot{3}$, $\cdot 1234\dot{5}\dot{6}$, $\cdot\dot{1}5\dot{7}$.

$$\begin{array}{r|r|l}
\cdot 5833 & 333333333333 & 3 \\
\cdot 0041 & 304130413041 & 3 \\
\cdot 1234 & 565656565656 & 5 \\
\cdot 1571 & 571571571571 & 5 \\
\hline
\cdot 8680 & 77469188360\!\!\!\!\;\;1 & \\
 & 1 \qquad\qquad\quad 2 &
\end{array}$$

Ans. $\cdot 8680\dot{7}7469188360\dot{2}$.

* If a number is divisible by two numbers prime to each other, it is divisible by their product; and, conversely, if a number is divisible by a product, it must be divisible by each factor. (Part I. Ch. XI.)

The first vertical line divides the non-recurring from the recurring figures.

Under the first recurring figure the carriage is noted down, to be added to the last figure of the period, as the next column to the right of the second vertical line would have given the same carriage.

Subtract $\cdot 7\dot{3}5\dot{8}$ from $4\cdot 372\dot{1}$.

$$\begin{array}{r|r|l} 4\cdot 372 & 111 & 1 \\ \cdot 735 & 835 & 8 \\ \hline \multicolumn{3}{l}{3\cdot 636\,27\cancel{6}} \\ \multicolumn{3}{l}{\quad\quad\;\; 1\;\;5} \end{array}$$

Ans. $3\cdot 636\dot{2}7\dot{5}$.

In this case the carriage is subtracted.

§ 17. In Part I. (p. 3 and Ch. X.) we have seen that we are not necessarily confined to any one scale of notation, but that we might have chosen any number whatsoever for our radix. Similarly in fractions we need not necessarily make the powers of 10 our universal denominator. The powers of any other number might have been chosen, and the resulting fractions would have become continuations of the corresponding integral scales. Thus,

Decimal	fractions are	a continuation of the	decimal or denary scale
Binal	"	"	binary scale,
Quinal	"	"	quinary scale,
Duodecimal	"	"	duodecimal scale, &c.

Reduce $\frac{7}{16}$ to senals.

$\frac{7}{16} = \frac{1}{16}$ of $7 = \frac{1}{16}$ of $\frac{42}{6} = \frac{2}{6}$ and $\frac{10}{6}$ over.

$\frac{1}{16}$ of $\frac{10}{6} = \frac{1}{16}$ of $\frac{60}{36} = \frac{3}{36}$ and $\frac{12}{36}$ over.

$\frac{1}{16}$ of $\frac{12}{36} = \frac{1}{16}$ of $\frac{72}{216} = \frac{4}{216}$ and $\frac{8}{216}$ over.

$\frac{1}{16}$ of $\frac{8}{216} = \frac{1}{16}$ of $\frac{48}{1296} = \frac{3}{1296}$.

$$Ans.\ \frac{2}{6} + \frac{3}{36} + \frac{4}{216} + \frac{3}{1296} = \frac{2}{6} + \frac{3}{6^2} + \frac{4}{6^3} + \frac{3}{6^4} = \cdot 2343.^{*}$$

Mod. op.: 7 and 16 in the senary scale = 11 and 24 respectively.

$$\begin{array}{rlllllll} & 6 & \text{I.} & \frac{1}{6} & \text{I.}\ \frac{1}{6} & \frac{1}{6^2} & \frac{1}{6^3} & \frac{1}{6^4} \\ 24) & 1 & 1\cdot & 0 & (\ \cdot 2 & 3 & 4 & 3 \\ & & 1 & 4 & 0 & & & \\ & & & 2 & 0\;\;0 & & & \\ & & & & 1\;\;2\;\;0 & & & \end{array}$$

* Read, "point" 2343, and not "decimal" 2343.

Reduce $\frac{1}{7}$ to binals.

$$\begin{array}{r@{\,}l} & \tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{8} \\ \text{Seven}\,) & \underline{1\cdot 0\ 0\ 0} \\ & \cdot 0\ 0\ 1\ldots \end{array} \qquad \textit{Ans. } \cdot\dot{0}0\dot{1}.$$

§ 18. We refrain from pursuing this subject further, but the curious are referred to Serret's Cours d'Arithmétique. We need only mention that the truths established for decimals hold, *mutatis mutandis*, for fractions in any scale, and that the several manipulations are identical. For example: the proof of Fermat's theorem given in § 6 holds for every scale of notation, substituting, however, for 999... repetition of the number which is 1 less than the radix.

§ 19. Summary.

Let $\frac{a}{b}$ be a fraction at lowest terms, and let it be reduced to a decimal.

(α) The character of the decimal will depend solely on b.

(β) If $b = 2^n$ or 5^n, or $2^n \times 5^m$, $\frac{a}{b}$ will terminate.

(γ) If $b = 2^n$ or 5^n, there will be n decimal places.

(δ) If $b = 2^n \times 5^m$, there will be n or m places according as n is greater or less than m.

(ϵ) If b is prime to 10, $\frac{a}{b}$ will give a pure circulator with a number of figures in the period, which number is invariable and depends solely on b.

(ζ) If b is a prime, this number is a measure of $b - 1$.

(η) If $b = 2^n \times c$, $5^n \times c$, or $2^n \times 5^m \times c$, c being prime to 10, the decimal will be a mixed circulator, having in the first and second cases n, and in the third n or m (whichever is greatest), non-recurring figures, followed by the number of recurring figures proper to c.

(θ) If b is prime to 10, and compounded of *different* primes, the number of recurring figures in the period is the L. C. M. of the numbers proper to these several primes.

(ι) If b is prime, and the number of figures in its period is even, the two halves of the period must be complemental to 9, and the corresponding remainders or dividends must be complemental to b.

CHAPTER VII.

DECIMALIZATION OF MONEY.

§ 1. We proceed to give a method of decimalizing our English money at sight, and to shew the numerous and great advantages accruing therefrom.

§ 2.

$$\begin{array}{lll} & 2s.\ (1\text{ florin}) & = £\cdot 1 \\ \therefore & 4s. & = £\cdot 2 \\ & 6s. & = £\cdot 3 \\ & \vdots & \quad\ \vdots \\ & 18s. & = £\cdot 9 \end{array}$$

$$\begin{array}{lllll} & 1s. = & \frac{1}{2}\text{ of }2s. & = \frac{1}{2}\text{ of }£\cdot 1 & = £\cdot 05 \\ \therefore & 3s. = & 2s + 1s. & = £\cdot 1 + £\cdot 05 & = £\cdot 15 \\ & 5s. = & 4s. + 1s. & = £\cdot 2 + £\cdot 05 & = £\cdot 25 \\ & 7s. = & 6s. + 1s. & = £\cdot 3 + £\cdot 05 & = £\cdot 35 \\ & \vdots & & & \vdots \\ & 19s. = & 18s. + 1s. & = £\cdot 9 + £\cdot 05 & = £\cdot 95 \end{array}$$

$$\begin{array}{llll} 6d. = & \frac{1}{2}\text{ of }1s. & = \frac{1}{2}\text{ of }£\cdot 05 & = £\cdot 025 \\ 1s.\ 6d. = & 1s. + 6d. & = £\cdot 05 + £\cdot 025 & = £\cdot 075 \\ 2s.\ 6d. = & 2s. + 6d. & = £\cdot 1 \ + £\cdot 025 & = £\cdot 125 \\ 3s.\ 6d. = & 2s. + 1s.\ 6d. & = £\cdot 1 \ + £\cdot 075 & = £\cdot 175 \\ \vdots & & & \vdots \\ 18s.\ 6d. = & 18s. + 6d. & = £\cdot 9 \ + £\cdot 025 & = £\cdot 925 \\ 19s.\ 6d. = & 18s. + 1s.\ 6d. & = £\cdot 9 \ + £\cdot 075 & = £\cdot 975 \end{array}$$

Rule: The figure in the first decimal place will indicate the number of florins; for an odd shilling, add 5 in the second place; for 6*d*. over, 25 in the second and third places; and for 1*s*. 6*d*. over, 75 in the second and third places.

EXERCISE XXXI.

Make a table of every sixpence from 6*d*. to 19*s*. 6*d*.

§ 3. THE ODD FARTHINGS.

6*d*. being £·025, $\frac{1}{4}d. = \cdot 025 \div 24$.

$$\begin{array}{r} 24)\ \underline{\cdot 025} \\ \cdot 001\frac{1}{24} \end{array}$$

which means: 1 farthing $= £\cdot001 + \frac{1}{24}$ of $£\cdot001$

$\therefore 1$ „ $= £\cdot001 + \frac{1}{12}$ of $\frac{1}{2}$ of $£\cdot001$*

2 farthings $= £\cdot002 + \frac{1}{12}$ of $\frac{1}{2}$ of $£\cdot002$

3 „ $= £\cdot003 + \frac{1}{12}$ of $\frac{1}{2}$ of $£\cdot003$

$\vdots \qquad \vdots$

$5\frac{1}{2}d. = 22$ „ $= £\cdot022 + \frac{1}{12}$ of $\frac{1}{2}$ of $£\cdot022$

$5\frac{3}{4}d. = 23$ „ $= £\cdot023 + \frac{1}{12}$ of $\frac{1}{2}$ of $£\cdot023$

Learn by heart: *Any number of farthings is the same number of thousandths of £1 + $\frac{1}{12}$ of $\frac{1}{2}$ of that number of thousandths.*

$$\begin{aligned} 1\tfrac{1}{2}d. = 6 \text{ f.} &= £\cdot006 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot006 \\ &= £\cdot006 + \tfrac{1}{12} \text{ of } £\cdot003 \\ &= £\cdot006 + £\cdot00025 \\ &= £\cdot00625 \end{aligned}$$

$$\begin{aligned} 4\tfrac{1}{2}d. = 18 \text{ f.} &= £\cdot018 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot018 \\ &= £\cdot018 + \tfrac{1}{12} \text{ of } £\cdot009 \\ &= £\cdot018 + £\cdot00075 \\ &= £\cdot01875 \end{aligned}$$

$$\begin{aligned} 3\tfrac{1}{2}d. = 14 \text{ f.} &= £\cdot014 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot014 \\ &= £\cdot014 + \tfrac{1}{12} \text{ of } £\cdot007 \\ &= £\cdot014 + £\cdot00058\dot{3} \\ &= £\cdot01458\dot{3} \end{aligned}$$

$$\begin{aligned} 1\tfrac{3}{4}d. = 7 \text{ f.} &= £\cdot007 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot007 \\ &= £\cdot007 + \tfrac{1}{12} \text{ of } £\cdot0035 \\ &= £\cdot007 + £\cdot000291\dot{6} \\ &= £\cdot007291\dot{6} \end{aligned}$$

$$\begin{aligned} 5\tfrac{1}{4}d. = 21 \text{ f.} &= £\cdot021 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot021 \\ &= £\cdot021 + \tfrac{1}{12} \text{ of } £\cdot0105 \\ &= £\cdot021 + £\cdot000875 \\ &= £\cdot021875 \end{aligned}$$

Observe: The half of an even number of thousandths is obvious; for an odd number, put on 5 in the next place to the half of the even number below it. The student must learn to write down the result at once, passing *mentally* through the steps indicated above.

* This might have been derived from the pound directly, thus:

1 farthing $= £\frac{1}{960}$; $£\frac{1}{960} > £\frac{1}{1000}$ by $£\frac{1}{960} - £\frac{1}{1000} = £\frac{1}{24000}$,

$\therefore 1$ „ $= £\cdot001 + \frac{1}{24}$ of $£\cdot001$.

This very useful expression for a farthing also supplies the very curious one: 1 farthing $= £\cdot001 + \cdot01d.$

Wording: 1½*d.* = 6 f. = £·0′0′6′; ½ of 6 = 3; 12 in 30, 2′, carry 6; in 60, 5′.
Ans. £·00625.

4½*d.* = 18 f. = £·0′1′8′; ½ of 18 = 9; 12 in 90, 7′; in 60, 5′. *Ans.* £·01875.

3½*d.* = 14 f. = £·0′1′4′; ½ of 14 = 7; 12 in 70, 5′; in 100, 8′; in 40, $\dot{3}$′.
Ans. £·01458$\dot{3}$.

1¾*d.* = 7 f. = £·0′0′7′; ½ of 7 = 35; 12 in 35, 2′; in 110, 9′; in 20, 1′; in 80, $\dot{6}$′.
Ans. £·007291$\dot{6}$.

5¼*d.* = 21 f. = £·0′2′1′; ½ of 21 = 105; 12 in 105, 8′; in 90, 7′; in 60, 5′.
Ans. £·021875.

¾*d.* = £·0′0′3′; ½ of 3 = 15; 12 in 15, 1′; in 30, 2′; in 60, 5′.
Ans. £·003125.

½*d.* = £·0′0′2′; ½ of 2 = 1; 12 in 10, 0′; in 100, 8′; in 40, $\dot{3}$′.
Ans. £·002083$\dot{3}$.

¼*d.* = £·0′0′1′; ½ of 1 = 5; 12 in 5, 0′; in 50, 4′; in 20, 1′; in 80, $\dot{6}$′.
Ans. £·001041$\dot{6}$.

4*d.* = 16 f. = £0′1′6′; ½ of 16, 8; 12 in 80, $\dot{6}$′. *Ans.* £01$\dot{6}$.

[6*d.* = 24 f. = £·0′2′4′; ½ of 24, 12; 12 in 12, 1′. *Ans.* £·025.]

Exercise XXXII.

Make a table of every farthing from ¼*d.* to 6*d.*

§ 4. Decimalize 15*s.* 8¼*d.*

15*s.* 6*d.*	= £·775
2¼*d.*	= £·009
	£·784 + $\frac{1}{12}$ of ½ of £·009 = £·784375.

The three places ·784 can be obtained at once by adding the 9 thousandths from the farthings to the 75 thousandths from the shillings, and we have then only to add $\frac{1}{12}$ of ½*d.* of ·009.

Wording: 15*s.* 6*d.* is ·7′75; 2¼*d.* is 9 f., and 75, 8′4′; ½ of 9 = 45; 12 in 45, 3′; in 90, 7′; in 60, 5′. *Ans.* £·784375.

Decimalize 13*s.* 10¾*d.*

Wording: 13*s.* 6*d.* = £·6′75; 4¾*d.* = 19 f., 9′4′; ½ of 19, 95; 12 in 95, 7′ in 110, 9′; in 20, 1′; in 8, $\dot{6}$′. *Ans.* £·694791$\dot{6}$.

EXERCISE XXXIII.

Decimalize:

(1) 8*s.* 6*d.*, 17*s.* 6*d.*, 13*s.* 6*d.*, 4*s.* 6*d.*, 1*s.* 6*d.*

(2) 13*s.* 3*d.*, 11*s.* 3*d.*, 19*s.* 3*d.*, 15*s.* 9*d.*, 18*s.* 9*d.*, 1*s.* 9*d.*

(3) 14*s.* $5\frac{1}{4}$*d.*, 11*s.* $10\frac{1}{2}$*d.*, 18*s.* $7\frac{1}{2}$*d.*, 3*s.* $11\frac{1}{4}$*d.*

(4) 8*d.*, 10*d.*, 7*d.*, $3\frac{1}{2}$*d.*, 1*s.* 1*d.*, 7*s.* 4*d.*

(5) 13*s.* 5*d.*, 17*s.* $10\frac{1}{4}$*d.*, 15*s.* $9\frac{1}{2}$*d.*, 13*s.* $8\frac{1}{2}$*d.*, 1*s.* $11\frac{1}{2}$*d.*, $\frac{1}{4}$*d.*, $\frac{1}{2}$*d.*, 1*d.*, 17*s.* $0\frac{1}{4}$*d.*, 11*s.* $0\frac{1}{2}$*d.*, 12*s.* $0\frac{1}{4}$*d.*

§ 5. RECONVERSION INTO MONEY.

1. The first decimal place gives florins.

2. A 5 (if any) in the second place gives 1*s.*

3. The remaining figures in the second and third places give each $\frac{24}{25}$ of a farthing; count them, then, as farthings, rejecting 1 if they exceed 24. The remaining figures of the decimal yield less than a farthing and may be disregarded.

Find the cost of 10, 100, 1000, 10000, 100000, and 1000000 articles, at 11*s.* $9\frac{3}{4}$*d.* each.

11*s.* $9\frac{3}{4}$*d.* = £·590625.

10 articles cost	£5·90625 = £5 + 9 fl. + 6 f. = £5. 18*s.* $1\frac{1}{2}$*d.*
100 ,,	£59.0625 = £59 + 1*s.* + 12 f. = £59. 1*s.* 3*d.*
1000 ,,	£590·625 = £590 + 6 fl. + 24 f. = £590. 12*s.* 6*d.*
10000 ,,	£5906·25 = £5906 + 2 fl. + 1*s.* = £5906. 5*s.*
100000 ,,	£59062·5 = £59062 + 5 fl. = £59062. 10*s.*
1000000 ,,	£590625.

Find the cost of 10, 100, 1000, &c., articles at 3*s.* $10\frac{1}{4}$*d.* each.

3*s.* $10\frac{1}{4}$*d.* = £·$192708\dot{3}$.

10 articles cost	£1·$92708\dot{3}$	= £1 + 9 fl. + 26 f. = £1. 18*s.* $6\frac{1}{4}$*d.*
100 ,,	£19·$2708\dot{3}$	= £19 + 2 fl. + 1*s.* + 20 f. = £19. 5*s.* 5*d.*
1000 ,,	£192·$708\dot{3}$	= £192 + 7 fl. + 8 f. = £192. 14*s.* 2*d.*
10000 ,,	£1927·$08\dot{3}$	= £1927 + 1*s.* + 32 f. = £1927. 1*s.* 8*d.*
100000 ,,	£19270·$83\dot{3}$*	= £19270 + 8 fl. + 32 f. = £19270. 16*s.* 8*d.*
1000000 ,,	£192708·$33\dot{3}$	= £192708 + 3 fl. + 32 f. = £192708. 6*s.* 8*d.*
10000000 ,,	£1927083·$33\dot{3}$	= £1927083 + 3fl. + 32f. = £1927083. 6*s.* 8*d.*

and so on.

* The student is advised always to form three places, adding ciphers where necessary.

Exercise XXXIV.

Read off the cost of 10, 100, 1000, 10000, 100000, 1000000 articles at:

(1) 8*s.* 6*d.*
(2) 9*s.* 5¼*d.*
(3) £2. 17*s.* 4½*d.*
(4) £3. 1*s.* 8¼*d.*
(5) 16*s.* 3¾*d.*
(6) £4. 13*s.* 3¾*d.*
(7) 2*s.* 7¼*d.*
(8) 3*s.* 7*d.*
(9) 4*s.* 8¾*d.*
(10) 5*s.* 10¾*d.*
(11) 6*s.* 4*d.*
(12) 1*s.* 5½*d.*
(13) 4*s.* 2*d.*
(14) 6*s.* 1*d.*
(15) 8*s.* 0¼*d.*
(16) 12*s.* 0½*d.*

The following values might with advantage be learned by heart:

$£\cdot\dot{3} = 6s.\ 8d.$
$£\cdot\dot{6} = 13s.\ 4d.$
$£\cdot0\dot{3} = 8d.$
$£\cdot0\dot{6} = 1s.\ 4d.$
$£\cdot01\dot{6} = 4d.$

$£\cdot01 = \frac{1}{5}$ of $£\cdot05 = \frac{1}{5}$ of $1s. = 2\frac{2}{5}d.$
$£\cdot02 = \frac{2}{5}$ of $1s. = 4\frac{4}{5}d.$
$£\cdot03 = 7\frac{1}{5}d.$
$£\cdot04 = 9\frac{3}{5}d.$

§ 6. Find the cost of 5307 articles at £2. 9*s.* 4¾*d.* each.

$£2.\ 9s.\ 4\frac{3}{4}d. = £2\cdot469791\dot{6}.$

```
 2·4697916    (3 places wanted.)
      7035
 ---------
  12348958
    740937
     17288
 ---------
 13107·183                 Ans. £13107. 3s. 8d.
```

Find the cost of $2473\frac{13}{25}$ articles at £6. 5*s.* 8*d.* each.

$2473\frac{13}{25} = 2473\cdot52.$ $£6.\ 5s.\ 8d. = £6\cdot28\dot{3}.$

```
  2473520      or      62833333
 33333826                253742
 --------             ---------
 14841120              12566667
   494704               2513333
   197882                439833
     7421                 18850
      742                  3142
       74                   126
        7             ---------
        1             15541·951
 --------
15541·951                    Ans. £15541. 19s. 0¼d.
```

Find the dividend on £537. 8s. 10d. at 9s. $7\frac{3}{4}$d. in the £.

£537·441$\dot{6}$ × ·482291$\dot{6}$ to 3 places.

$$\begin{array}{r} 537{\cdot}442 \\ 2922\ 84 \\ \hline 21\ 4977 \\ 4\ 2995 \\ 1075 \\ 108 \\ 48 \\ 1 \\ \hline 259{\cdot}204 \end{array}$$

Ans. £259. 4s. 1d.

Exercise XXXV.

I.

(1) 3562 articles at 15s. $8\frac{3}{4}$d. each.
(2) 6019 „ £1. 2s. 10d. each.
(3) 7038 „ £3. 14s. $2\frac{3}{4}$d. each.
(4) 269 „ 3s. $10\frac{3}{4}$d. each.
(5) 5966 „ £7. 16s. 8d. each.
(6) 2469 „ £1. 10s. $10\frac{1}{4}$d. each.
(7) 9004 „ £2. 7s. $6\frac{1}{2}$d. each.
(8) 5040 „ 7s. $11\frac{1}{2}$d. each.
(9) 10010 „ 1s. $8\frac{3}{4}$d. each.
(10) 5039 „ 3s. $7\frac{1}{2}$d. each.

II.

(1) $843\frac{4}{5}$ articles at £3. 16s. $8\frac{3}{4}$d. each.
(2) $2047\frac{2}{3}$ „ £1. 17s. $2\frac{1}{2}$d. each.
(3) $3195\frac{5}{6}$ „ £2. 2s. $2\frac{1}{2}$d. each.
(4) 9843 doz. and 5 articles at 3s. $2\frac{1}{4}$d. per dozen.
(5) 7054 doz. and 11 „ £1. 9s. 10d. per dozen.
(6) 2437 doz. and 7 „ £2. 13s. $9\frac{1}{2}$d. per dozen.
(7) 5 years and 7 months at £31. 11s. 6d. a year.
(8) 10 years and 5 months at £9. 17s. 3d. a year.
(9) 17 years and 11 months at £10. 10s. 10d. a year.
(10) £1870. 16s. 8d. at 7s. $9\frac{3}{4}$d. in the £.
(11) £497. 10s. 4d. at 16s. $7\frac{1}{4}$d. in the £.

III.

(1) $2473\frac{13}{25}$ articles at £6. 5*s.* 8*d.* each.
(2) $649\frac{5}{7}$ „ £1. 16*s.* 2*d.* each.
(3) $4037\frac{3}{11}$ „ £2. 11*s.* $8\frac{1}{2}$*d.* each.
(4) $953\frac{13}{18}$ „ £8. 7*s.* 10*d.* each.
(5) $2583\frac{7}{16}$ „ £3. 3*s.* $11\frac{1}{2}$*d.* each.
(6) $7211\frac{7}{10}$ „ £1. 18*s.* $7\frac{3}{4}$*d.* each.
(7) $2045\frac{4}{17}$ „ £2. 2*s.* $9\frac{1}{2}$*d.* each.
(8) $477\frac{43}{78}$ „ £4. 9*s.* 8*d.* each.
(9) 700700·07 „ £10. 5*s.* 7*d.* each.
(10) $843594\frac{3}{4}$ „ $6\frac{1}{2}$*d.* each.

§ 7. Decimalization of Other Fractions of a Penny.

Case I. Decimalize $9\frac{5}{8}d.$

$9\frac{5}{8}d. \times 2 = 1s.\ 7\frac{1}{4}d. = £\cdot 080208\dot{3}$, $£\cdot 080208\dot{3} \div 2 = £\cdot 0401041\dot{6}$.

Ans. $£\cdot 0401041\dot{6}$.

Case II. Decimalize $3\frac{11}{16}d.$

$3\frac{11}{16}d. \times 4 = 1s.\ 2\frac{3}{4}d. = £\cdot 061458\dot{3}$, $£\cdot 061458\dot{3} \div 4 = £\cdot 015364458\dot{3}$.

Ans. $£\cdot 015364458\dot{3}$.

Case III. Decimalize $2\frac{1}{32}d.$

$2\frac{1}{32}d. \times 8 = 1s.\ 4\frac{1}{4}d. = £\cdot 067708\dot{3}$, $£\cdot 067708\dot{3} \div 8 = £\cdot 008463541\dot{6}$.

Ans. $£\cdot 008463541\dot{6}$.

Case IV. Decimalize $7\frac{3}{10}d.$

$7\frac{3}{10}d. \times 10 = 6s.\ 1d. = £\cdot 3041\dot{6}$, $£\cdot 3041\dot{6} \div 10 = £\cdot 03041\dot{6}$.

Ans. $£\cdot 03041\dot{6}$.

Case V. Decimalize $5\frac{3}{5}d.$

$5\frac{3}{5}d. \times 5 = 2s.\ 4d. = £\cdot 11\dot{6}$, $£\cdot 11\dot{6} \div 5 = £\cdot 02\dot{3}$. *Ans.* $£\cdot 02\dot{3}$.

or,

$5\frac{3}{5}d. = 5\cdot 6d.$, $5\cdot 6d. \times 10 = 56d. = 4s.\ 8d. = £\cdot 2\dot{3}$, $£\cdot 2\dot{3} \div 10 = £\cdot 02\dot{3}$.

Ans. $£\cdot 02\dot{3}$.

Case VI. Decimalize $7\frac{13}{20}d.$

$7\frac{13}{20}d. \times 10 = 6s.\ 4\frac{1}{2}d. = £\cdot 31875$; $£\cdot 31875 \div 10 = £\cdot 031875$.

Ans. $£\cdot 031875$.

Case VII. Decimalize $11\frac{4}{7}d.$

$11\frac{4}{7}d. \times 7 = 6s.\ 9d. = £\cdot 3375$, $£\cdot 3375 \div 7 = £\cdot 048\dot{2}14285\dot{7}$.

Ans. $£\cdot 048\dot{2}14285\dot{7}$.

Cases I., II. and III., are of common occurrence in commerce. Cases IV., V. and VI., are easy of calculation; for Case V. two methods are given, of which the second can be done mentally. Case VII. and the like are only to be found in examination papers.

Exercise XXXVI.

(1) 28493 lbs. of raw cotton at $9\frac{11}{16}d.$
(2) 97058 „ $1s.\ 0\frac{5}{8}d.$
(3) 247963 „ $11\frac{19}{32}d.$
(4) 519766 lbs. of waste cotton at $1\frac{25}{64}d.$
(5) 27964 „ $3\frac{1}{5}d.$
(6) 24572 „ $4\frac{4}{7}d.$
(7) 24572 „ $2\frac{2}{5}d.$
(8) 967458 „ $\frac{53}{64}d.$
(9) 1000000 „ $1\frac{3}{7}d.$
(10) 257683 „ $2\frac{7}{10}d.$
(11) 489573 „ $4\frac{13}{20}d.$
(12) 359087 „ $1s.\ 1\frac{11}{20}d.$

§ 8. Division of Money by Money.

In every case that actually occurs, the recurring figure in decimalized money will be 3 or 6,* which can be got rid of by multiplying by 3, for $3 \times \cdot\dot{3} = \cdot\dot{9} = 1$, $\therefore 3 \times \cdot\dot{6} = 2$.

Case I. £74. 8s. $4\frac{1}{2}d. \div$ £1 9s. $6\frac{3}{4}d.$ to 5 places.

$$
\begin{array}{r}
1{\cdot}4\cancel{7}\cancel{8}\cancel{1}\cancel{2}\cancel{5})\,74{\cdot}41875(50{\cdot}34672 \\
{\cdot}512500 \\
69062 \\
9937 \\
1068 \\
83
\end{array}
$$

Ans. 50·34672.

or, stopping at the integral part of the quotient:

Ans. 50 times and £·5125 = 10s. 3d. over.

* 6d. = £·025.
$\frac{3}{4}d. = \frac{1}{8}$ of 6d., terminating (Ch. VI. § 2).
$\frac{1}{4}d. = \frac{1}{3}$ of $\frac{3}{4}d.$, recurring, ∵ ·025 is not divisible by 3.
$\frac{1}{2}d. = \frac{2}{3}$ of $\frac{3}{4}d.$ „ „

If, then, the number of farthings over 6d. be divisible by 3, the decimal will terminate; if, when divided by 3, there be a remainder 1, the recurring figure is $\dot{3}$; if the remainder be 2, the recurring figure is $\dot{6}$.

CASE II. £58. 13*s*. 7*d*. ÷ 13*s*. $10\frac{1}{4}$*d*.

$$£58\cdot 6791\dot{6} \div £\cdot 692708\dot{3} = £58\cdot 6791\dot{6} \times 3 \div £\cdot 692708\dot{3} \times 3$$
$$= £176\cdot 0375 \div £2\cdot 078125.$$

```
2·078125)176·0375(84
           9·7875
         3)1·4750
           ·491̇6
```

Ans. 84 times and 9*s*. 10*d*. over.

The remainder is divided by 3, in accordance with Ch. III. § 8, Case III.; or this may be done by contracted division at once, without multiplication by 3. (Ch. IV. p. 147, D.)

If in the divisor the fractional part of the penny be other than farthings, the denominator must be reduced by multiplication as in § 7.

EXERCISE XXXVII.

Work by decimals Exercise XX. in Part I.

CHAPTER VIII.

DECIMALIZATION OF WEIGHTS AND MEASURES.

§ 1. AVOIRDUPOIS WEIGHT.

a. Tons and cwts. at per ton.

1 ton : 1 cwt. = £1 : 1*s*. Hence call the cwts. shillings, decimalize and multiply.

Find the cost of 57 tons, 13 cwt., at £1. 5*s*. 9*d*. per ton.

57 tons, 13 cwt. (£57. 13*s*. = £57·65) = 57·65 tons. £1. 5*s*. 9*d*. = £1·2875.

```
 12875          or      12·875
  5765                    5675
 ------                 ------
 64375                   64375
  90125                   9013
   77250                   772
    64375                   64
 ---------              ------
 74·224375              74·224
```

Ans. £74. 4*s*. 6*d*.

b. Tons, cwts., qrs., at per ton.

1 ton : 1 qr. = £1 : 3*d.* Hence call every quarter 3*d.*, &c.

Find the cost of 17 tons, 11 cwt., 3 qrs., at £5. 2*s.* 6*d.* per ton.

17 tons, 11 cwt., 3 qrs. (£17. 11*s.* 9*d.*) = 17·5875 tons. £5. 2*s.* 6*d.* = £5·125.

$$\begin{array}{r} 17{\cdot}5875 \\ 5\;215 \\ \hline 87\;938 \\ 1\;759 \\ 352 \\ 88 \\ \hline 90{\cdot}137 \end{array}$$

Ans. £90. 2*s.* 9*d.*

c. Tons, cwts., qrs., lbs., at per ton, the lbs. being a multiple of 7 lbs.

1 ton : 7 lbs. = £1 : $\frac{3}{4}$*d.* Hence call every 7 lbs. $\frac{3}{4}$*d.*

Find the cost of 19 tons, 5 cwt., 1 qr., 14 lbs., at 18*s.* 10$\frac{1}{2}$*d.* per ton.

19 tons, 5 cwt., 1 qr., 14 lbs. (£19. 5*s.* 4$\frac{1}{2}$*d.*) = 19·26875 tons.
18*s.* 10$\frac{1}{2}$*d.* = £·94375.

$$\begin{array}{r} 19{\cdot}268\cancel{7}\cancel{5} \\ 573\;49 \\ \hline 173\;41 \\ \dot{7}\;70 \\ 58 \\ 13 \\ 1 \\ \hline 18{\cdot}183 \end{array}$$

Ans. £18. 3*s.* 8*d.*

Exercise XXXVIII.

Find the cost of:

(1) 43 tons, 17 cwt., at £5. 8*s.* 3*d.* per ton.

(2) 457 tons, 9 cwt., at £7. 10*s.* 10*d.* per ton.

(3) 8 tons, 1 cwt., at £1. 9*s.* 4$\frac{1}{2}$*d.* per ton.

(4) 9 tons, 8 cwt., 1 qr., at 16*s.* 8$\frac{3}{4}$*d.* per ton.

(5) 6 tons, 13 cwt., 2 qrs., at 17*s.* 9*d.* per ton.

(6) 7 tons, 5 cwt., 3 qrs., at 12*s.* 5*d.* per ton.

(7) 4 tons, 4 cwt., 1 qr., 7 lbs., at £43. 12*s.* 6*d.* per ton.

(8) 12 tons, 2 cwt., 2 qrs., 14 lbs., at £52. 10*s.* per ton.

(9) 23 tons, 3 cwt., 3 qrs., 21 lbs., at £24. 10*s.* 10*d.* per ton.

(10) 16 cwt., 1 qr., 14 lbs., at £18. 17*s.* 4*d.* per ton.

d. Tons, cwts., qrs., lbs., at per ton.

7 lbs. ($\frac{3}{4}d.$) = ·003125 tons.

1 lb. = $\frac{1}{7}$ of 7 lbs. = $\frac{1}{7}$ of ·003125 tons.

2 lbs. = $\frac{1}{7}$ of 2 × 7 lbs. = ($\frac{1}{7}$ of 2 × $\frac{3}{4}$ = $\frac{1}{7}$ of $1\frac{1}{2}d.$) = $\frac{1}{7}$ of ·00625 tons.

5 lbs. ($\frac{1}{7}$ of $3\frac{3}{4}d.$) = $\frac{1}{7}$ of ·015625 tons.

&c. &c.

Reduce 4 lbs. to the decimal of a ton.

4 lbs. ($\frac{1}{7}$ of 3*d.*) = $\frac{1}{7}$ of ·0125 = $\cdot 0017\dot{8}5714\dot{2}$.

Wording: 7 in 0, 0′; in 1, 0′; in 12, 1′; in 55, 7′, and 6 over; $\frac{6}{7} = \cdot\dot{8}5714\dot{2}$. (See ring, Ch. VI. § 8.)

Express as tons, 13 cwt., 1 qr., 24 lbs.

13 cwt., 1 qr., 21 lbs. (13*s.* $5\frac{1}{4}d.$) = ·671875

3 lbs. ($\frac{1}{7}$ of $2\frac{1}{4}d.$) = $\frac{1}{7}$ of £·009375 = $\cdot 001339\dot{2}8571\dot{4}$

$\cdot 673214\dot{2}8571\dot{4}$

Ans. $\cdot 6732\dot{1}4285\dot{7}$ tons.

Find the cost of 5 tons, 7 cwt., 1 qr., 13 lbs., at £4. 7*s.* $8\frac{3}{4}d.$ per ton.

5 tons, 7 cwt., 1 qr., 7 lbs. (£5. 7*s.* $3\frac{3}{4}d.$) = 5·365625

6 lbs. ($\frac{1}{7}$ of $4\frac{1}{2}d.$) = $\frac{1}{7}$ of ·01875 = ·002678, &c.

5·368303 tons.

£4. 7*s.* $8\frac{3}{4}d.$ = $4\cdot 386458\dot{3}$.

5·368303

56 834

21 473

1 610

429

32

3

23·547 *Ans.* £23. 10*s.* $11\frac{1}{4}d.$

It will be observed that we have carried out the decimal further than was necessary. With a little practice the student will learn *to avoid* all such unnecessary labour.

Find the cost of 149 tons, 13 cwt., 3 qrs., 10 lbs., at £43. 8*s.* 4*d.* per ton.

149 tons, 13 cwt., 3 qrs., 7 lbs. (£149. 13*s.* 9$\frac{3}{4}$*d.*) = 149·690625
3 lbs. ($\frac{3}{7}$ of 2$\frac{1}{4}$*d.*) = $\frac{3}{7}$ of ·009375 = ·001339
149·691964 tons.

£43. 8*s.* 4*d.* = £43·41$\dot{6}$.

$$\begin{array}{r} 149\cdot69196 \\ \underline{6666\ 1434} \\ 598\ 7678 \\ 44\ 9076 \\ 5\ 9876 \\ 1497 \\ 898 \\ 90 \\ 9 \\ 1 \\ \hline 6499\cdot125^{*} \end{array}$$

Ans. £6499. 2*s.* 6*d.*

Exercise XXXIX.

Find the cost of:

(1) 18 tons, 9 cwt., 1 qr., 16 lbs., at £3. 10*s.* 4*d.* per ton.
(2) 23 tons, 0 cwt., 2 qrs., 20 lbs., at £5. 4*s.* 7$\frac{1}{2}$*d.* per ton.
(3) 93 tons, 7 cwt., 3 qrs., 8 lbs., at £6. 15*s.* 8*d.* per ton.
(4) 7 tons, 5 cwt., 1 qr., 21 lbs., at £10. 4*s.* per ton.
(5) 86 tons, 11 cwt., 2 qrs., 11 lbs., at £51. 17*s.* 4$\frac{1}{2}$*d.* per ton.
(6) 17 cwt., 3 qrs., 26 lbs., at £9. 12*s.* 10*d.* per ton.
(7) 3 tons, 3 cwt., 3 qrs., 3 lbs., at £3. 3*s.* 3*d.* per ton.
(8) 2 tons, 14 cwt., 1 qr. 7 lbs., at £2. 14*s.* 3$\frac{3}{4}$*d.* per ton.
(9) 5 cwt., 9 lbs., at £16. 16*s.* per ton.
(10) 4 tons, 1 qr., at £2. 5*s.* 8*d.* per ton.

e. cwts., qrs., and lbs., at per cwt.

1 cwt. : 1 qr. = £1 : 5*s.*, ∴ 1 qr. = ·25 cwt.
1 cwt. : 7 lbs. = £1 : 1*s.* 3*d.*, ∴ 7 lbs. = ·0625 cwt.
1 cwt. : 1 lb. = £1 : $\frac{1}{7}$ of 1*s.* 3*d.*, ∴ 1 lb. = ·0089$\dot{2}$8571$\dot{4}$.

* Comparison with the mode of working the same sum by Practice, Part II. p. 76, wili justify the foot-note of p. 75. The difference in the number of figures used gives but a slight indication of the saving of ingenuity and labour, uniform simple multiplication being substituted for the intricate manipulation of fractions. The correctness of the work admits of an easy test by transposing the multiplier and multiplicand.

Find the cost of 3 tons, 13 cwt., 2 qrs., 19 lbs., at £1. 6s. 8d. per cwt.

73 cwt., 2 qrs., 14 lbs. (£73. 12s. 6d.) $= 73{\cdot}625$ cwt.
5 lbs. ($\frac{1}{7}$ of $5 \times 1s.\ 3d. = \frac{1}{7}$ of $6s.\ 3d.) = \frac{1}{7}$ of $\cdot 3125 = \ \cdot 044\dot{6}42857\dot{1}$

73·6696, &c.

£1. 6s. 8d. = £1·$\dot{3}$ or £$\frac{4}{3}$.

73·6696		
333·331		
73 670	or	73·6696
22 101		4
2 210		3)294·678
221		98·226
22		
2		
98·226		

Ans. **£98. 4s. 6½d.**

Exercise XL.

Find the cost of:

(1) 4 cwt., 1 qr., 17 lbs., at £3. 3s. per cwt.
(2) 3 qrs., 19 lbs., at £7. 8s. 4d. per cwt.
(3) 1 ton, 7 cwt., 3 qrs., 24 lbs., at £10. 12s. 8d. per cwt.
(4)*3 cwt., 2 qrs., 16 lbs., at £3. 7s. 8d. per cwt.
(5) 9 tons, 5 cwt., 16 lbs., at 12s. per cwt.
(6) 45 tons, 1 qr., 9 lbs., at £2. 7s. 10d. per cwt.
(7) 3 tons, 2 qrs., 8 lbs., at £5. 7s. 4d. per cwt.
(8) 3 cwt., 3 qrs., 12 lbs., at £1. 4s. 11d. per cwt.
(9) 15 tons, 15 cwt., 1 qr., 20 lb., at £10. 10s. per cwt.

f. lbs. and oz. at per lb.
1 lb. : 1 oz. = £1 : 1s. 3d., ∴ 1 oz. = ·0625 lbs.

Ounces being binary fractions of the lb., are very easily decimalized, every two ounces being called 2s. 6d.

* N.B. 16 lbs. $= \frac{1}{7}$ of 1 cwt. ; 2 qrs., 8 lbs. $= \frac{4}{7}$ of 1 cwt.
1 qr., 4 lbs. $= \frac{2}{7}$ of 1 cwt. ; 2 qrs., 24 lbs. $= \frac{5}{7}$ of 1 cwt.
1 qr., 20 lbs. $= \frac{3}{7}$ of 1 cwt. ; 3 qrs., 12 lbs. $= \frac{6}{7}$ of 1 cwt.
all of which can be decimalized at once from the ring, p. 161.

Express 11 oz. as a decimal of 1 lb.

11 oz. (11 × 1*s*. 3*d*. = 13*s*. 9*d*.) = ·6875 lbs.

Also 12 oz.

12 oz. (6 × 2*s*. 6*d*.) = ·75 lbs.

Exercise XLI.

Find the cost of:

(1) 3 qrs., 17 lbs., 5 oz., at 3*s*. 8*d*. per lb.

(2) 4 lbs., 14 oz., at £1. 17*s*. 6*d*. per lb.

(3) 9 oz., at £2. 15*s*. 4*d*. per lb.

(4) 15 lbs., 15 oz., at £6. 4*s*. $10\frac{1}{2}d$. per lb.

(5) 1 cwt., 10 lbs., 6 oz., at 18*s*. $7\frac{1}{2}d$. per lb.

(6) 9 lbs., 8 oz., at £2. 14*s*. 11*d*. per lb.

§ 2. Troy Weight.

a. lbs., oz., dwts. at per lb.
1 lb. : 1 oz. = £1 : 1*s*. 8*d*.
1 lb. : 1 dwt. = £1 : 1*d*. (Hence the name "*penny* weight.")
Hence call each oz. 1*s*. 8*d*., and each dwt. 1*d*.

Find the cost of 9lbs., 3 oz., 14 dwt., at £10. 15*s*. 6*d*. per lb.

9 lbs., 3 oz., 14 dwt. (£9. 6*s*. 2*d*.) = $9{\cdot}308\dot{3}$ lbs. £10. 15*s*. 6*d*. = £10·775.

```
 9·30833
 5 7701
 -------
 9 3083
   6516
    651
     47
 -------
100·297
```

Ans. £100. 5*s*. $11\frac{1}{4}d$.

As the lb. troy is hardly ever used, we proceed at once to the next case.

b. oz., dwts., grs., at per oz.
1 oz. : 1 dwt. = £1 : 1*s*.
1 oz. : 1 gr. = £1 : $\frac{1}{2}d$.
Hence call each dwt. 1*s*., and each gr. $\frac{1}{2}d$.

Find the cost of a gold snuff-box weighing 7 oz., 15 dwts., 15 grs., at £4. 5*s.* 6*d.* per oz.

7 oz., 15 dwts., 15 grs. (£7. 15*s.* 7½*d.*)=7·78125 oz. £4. 5*s.* 6*d.* =£4·275.

```
 7·78125
 5 724
--------
31 125
 1 556
   545
    39
--------
33·265
```

Ans. £33. 5*s.* 3¾*d.*

Exercise XLII.

(1) Find the cost of 17 lbs., 9 oz., 15 dwts., at £8. 4*s.* 11*d.* per lb.
(2) " 9 lbs., 6 oz., 19 dwts., at £4. 19*s.* 4½*d.* per lb.
(3) " 1 lb., 7 oz., 15 dwts., 20 grs. at £3. 17*s.* 10½*d.* per oz.
(4) " 8 oz., 11 dwts., 17 grs., at £1. 14*s.* 10*d.* per oz.
(5) " 10 oz., 10 dwts., 10½ grs., at £7. 11*s.* 4*d.* per oz.
(6) " 4 lbs., 11 oz., 11 dwts., 11½ grs. at £10. 10*s.* per oz.

§ 3. Capacity.

a. gals., qts., pts. and gills, at per gal.

1 gal. : 1 qt. = £1 : 5*s.*
1 " : 1 pt. = £1 : 2*s.* 6*d.*
1 " : 1 gill = £1 : 7½*d.*

Hence call each quart 5*s.*, each pint 2*s.* 6*d.*, and each gill 7½*d.*

Find the cost of 5 gals., 3 qts., 1 pt., 1 gill, at £1. 5*s.* 6*d.* per gal.

5 gals., 3 qts., 1 pt., 1 gill (£5. 18*s.* 1½*d.*)=5·90625 gals. £1. 5*s.* 6*d.* =£1·275.

```
5·90625
5 721
-------
5 906
1 181
  413
   30
-------
7·530
```

Ans. £7. 10*s.* 7¼*d.*

b. Quarters, bus., pecks, at per quarter.

1 qr. : 1 bus. = £1 : 2*s.* 6*d.*

1 qr. : 1 peck = £1 : $7\frac{1}{2}$*d.*

Hence call each bushel 2*s.* 6*d.*, and each peck $7\frac{1}{2}$*d.*

Find the cost of 43 qrs., 5 bus., 3 pecks, at £2. 10*s.* per qr.

43 qrs., 5 bus., 3 pks. (£43. 14*s.* $4\frac{1}{2}$*d.*) = 43·71875 qrs.
£2. 10*s.* = £2·5

4)43·71875

109·296, &c. = £109. 5*s.* $11\frac{1}{4}$*d.*

Exercise XLIII.

(1) Find the cost of 15 gals., 3 qts., 1 pt., 3 gills, at $10\frac{1}{2}$*d.* per gal.

(2) „ 2 qts., 1 pt., 1 gill, at 1*s.* $4\frac{3}{4}$*d.* per gal.

(3) „ 29 gals., 1 qt., 1 pt., 2 gills, at 2*s.* $10\frac{1}{2}$*d.* per gal.

(4) „ 428 qrs., 3 bus., 2 pecks, at £2. 13*s.* 6*d.* per qr.

(5) „ 617 qrs., 1 bus., 3 pecks, at £1. 3*s.* 9*d.* per qr.

§ 4. Length.

a. Yds., ft. and in., at per yard and per foot.

It is shortest to reduce the yards into feet and decimalize the inches by dividing by 12. If the price is given per ft., the decimal is at once available; if per yd., divide either the decimal or the price or the ultimate product by 3.

Find the cost of 8 yds., 1 ft., 11 in., at £1. 4*s.* 7*d.* per yd.

8 yds., 1 ft., 11 in. = $25{\cdot}91\dot{6}$ ft. = $8{\cdot}63\dot{8}$ yds. £1. 4*s.* 7*d.* = £1·2291$\dot{6}$

```
 8·6388
29 221
-------
 8 639
 1 727
   173
    77
     2
-------
10·618
```

Ans. £10. 12*s.* $4\frac{1}{4}$*d.*

b. Miles, fur. and yds., at per mile.

1 m. : 1 fur. = £1 : 2*s.* 6*d.*

1 m. : 11 yds. = £1 : $1\frac{1}{2}$*d.*

Hence call each furlong 2*s.* 6*d.*, each 11 yds. $1\frac{1}{2}$*d.*, and each yd. $\frac{1}{11}$ of $1\frac{1}{2}$*d.*

Find the cost of 2 miles, 3 fur., 105 yds., at £15. 15*s.* per mile.

2 m., 3 fur., 99 yds. (£2. 8*s.* $7\frac{1}{2}$*d.*) = 2·43125 m.
6 yds. ($\frac{1}{11}$ of $6 \times 1\frac{1}{2}$*d.*) = $\frac{1}{11}$ of ·0375 = $\cdot 0034\dot{0}\dot{9}$

$2{\cdot}434659\dot{0}\dot{9}$ m.

£15. 15*s.* = £15·75.

```
2·434659
  5751
--------
2 4347
1 2173
  1704
   122
--------
38·346
```

Ans. £38. 6*s.* $11\frac{1}{4}$*d.*

c. Gunter's chain.

5 chains, $47\frac{1}{2}$ links, at 7*s.* $8\frac{1}{2}$*d.* per ch.

$5{\cdot}475 \times {\cdot}38541\dot{6}$

```
 3854
 5745
------
 1927
  154
   27
    2
------
2·110
```

Ans. £2. 2*s.* $2\frac{1}{2}$*d.*

Exercise XLIV.

(1) Find the value of 7 chains, $3\frac{3}{4}$ links, at 8*s.* $10\frac{1}{2}$*d.* per chain.
(2) „ 49 chains, $18\frac{1}{2}$ links, at 16*s.* $5\frac{1}{4}$*d.* per chain.
(3) „ 356 chains, $59\frac{1}{4}$ links, at £2. 14*s.* 8*d.* per chain.
(4) „ 315 miles, 6 fur., at £768. 15*s.* per mile.
(5) „ 19 yds., 2 ft., 8 in., at 7*s.* $8\frac{1}{2}$*d.* per yd.
(6) „ 43 yds., 1 ft., 11 in., at 2*s.* $5\frac{3}{4}$*d.* per yd.
(7) „ 1 mile, 4 fur., 77 yds., at £100 per mile.
(8) „ 5 miles, 5 fur., 50 yds., at £20. 8*s.* per mile.

§ 5. SURFACE.

Acres, roods, poles, at per acre.

1 a. : 1 r. = £1 : 5*s.*
1 a. : 1 p. – £1 : $1\frac{1}{2}d.$

Hence call each rood 5*s.*, and each pole $1\frac{1}{2}d.$

Find the cost of 18 acres, 2 roods, 19 poles, at £46. 15*s.* per acre.

18 acres, 2 roods, 19 poles (£18. 12*s.* $4\frac{1}{2}d.$) = 18·61875 acres.

£46. 15*s.* = £46·75.

```
18·61875
   5764
--------
74 4750
11 1712
 1 3033
    931
--------
870·426
```

Ans. £870. 8*s.* $6\frac{1}{4}d.$

EXERCISE XLV.

(1) Find the cost of a close of 37 acres, 3 roods, 25 poles, at £42. 10*s.* per acre.

(2) „ field of 19 acres, 1 rood, 35 poles, at £57. 15*s.* per acre.

(3) „ farm of 368 acres, 2 roods, 30 poles, at £41. 15*s.* per acre.

(4) I bought 1572 acres, $1\frac{1}{2}$ roods, at £37. 17*s.* 6*d.* per acre, and of it sold of the best land 419 acres, $3\frac{1}{2}$ roods, at £47. 5*s.* an acre; how much an acre did the remainder stand me in?

§ 6. This method of translation into money would also be applicable to other weights and measures. Thus in paper measure, calling 1 ream £1, the quire is 1*s.*, and the sheet $\frac{1}{2}d.$; but we leave all further applications to the ingenuity of the student.

CHAPTER IX.

DECIMALS APPLIED TO PROPORTION.

Find the value of 37 yards of silk, when 25 yards cost £4. 7s. 6d.

$\cancel{25} : 37 = 4{\cdot}375 : x$
* (×4) 100   4
4·375
4
17·5$\cancel{00}$
37
1225
525
6·475   *Ans.* £6. 9s. 6d.

If a workman earns £17. 6s. in $102\frac{1}{2}$ days, how long will he be in earning 50 guineas?

$17{\cdot}3 : 52{\cdot}5 = 102{\cdot}5 : x$
102·5
52 5
512 5
2050
5125
173)53812·5(311
191
182
9·5
$\overline{173} = \frac{95}{1730} \big| \frac{19}{346}$.   *Ans.* $311\frac{19}{346}$ days.

If the tax on £195 be £14. 8s., what will be the tax on £874?

$\cancel{195} : £874 = \cancel{14}{\cdot}\cancel{4} : x$
65   4·8
874
6
5244
8
6$\cancel{5}$) 4195·2 (64·541
295
352
270
10   *Ans.* £64. 10s. 10d.

* Since $\frac{1}{4} = {\cdot}25$ and $\frac{3}{4} = {\cdot}75$, decimal fractions ending in 25 or 75 are often conveniently reduced to lower terms by multiplication by 4; similarly, decimals terminating in 125, 375, 625 and 875, are reducible by multiplication by 8. (Cf. *Part. I. Ch. IX.* § 19.)

If I can travel 198 miles by railway for £2. 9*s.*, how far at the same rate of charge ought I to be carried for £8. 0*s.* $10\frac{1}{2}d$?

$$2{\cdot}475 : 8{\cdot}04375 = 198 : x$$

(×8) 19·8   8

·1   64·35000

643·5

Ans. 643·5 miles.

The annual poor's-rates on a nett rental of £365. 7*s.* 3*d.* amount to £36. 8*s.* 9*d.* What should be the nett rental of an estate for which the poor's-rates amount to £24. 5*s.* 10*d.*?

$$36{\cdot}4375 : 24{\cdot}291\dot{6} = 365{\cdot}3625 : x$$

36·4375	24·2916̇	365·3625
8	3	8
291·5000	72·875	2922·9000
3	8	58 3
874·5	583·000	8768 7
8		233832
6996		146145
		1704050·7

6996) 1704050·7 (243·575
30485
25010
40227
52470
34980
·····

Ans. £243. 11*s.* 6*d.*

Exercise XLVI.

Work, Part II., Ex. LI., Nos. 4, 7, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 53, 54, 63.

CHAPTER X.

PERCENTAGES.

§ 1. In commerce, the universal standard of comparison for estimating Profit and Loss, Premiums, Interest, Commission, Brokerage,

&c., is 100, whence the term *per centum*, shortened into *per cent.*, for which the symbol is %. Thus :

A profit of £5 on an *outlay* of £100 is called a profit of 5 %; similarly, a profit of 5*s.* on an outlay of 100*s.* is a profit of 5 %.

A premium of £2. 3*s.* 4*d.* to insure against the loss of £100 is called a premium of £2. 3*s.* 4*d.* %, or $2\frac{1}{6}$ %, or $2{\cdot}1\dot{6}$ %.

A brokerage of $\frac{1}{8}$ % means that the broker is to receive 2*s.* 6*d.* on every £100 transferred through him, &c.

Find the commission on £360 at 2 %.

$$\begin{array}{rcl} \text{If on £100} & \text{we pay} & £2, \\ \text{on £360} & ,, & £x. \end{array}$$

$$100 : 360 = 2 : x$$
$$3{\cdot}6 \times 2 = 7{\cdot}2.$$

Ans. £7. 4*s.*

Find the payment on £p at r %.

$$\begin{array}{rcl} \text{If on £100} & \text{we pay} & £r, \\ \text{on £}p & ,, & £x. \end{array}$$

$$100 : p = r : x$$
$$x = \frac{p \times r}{100}$$

or, if we call the payment £P, we obtain the "formula,"

$$* \; P = \frac{p \times r}{100} \qquad \text{(I.)}$$

What is the sum to be paid for insuring a vessel and cargo worth £2225, at $3\frac{1}{4}$ %?

Here $p = £2225$, $r = 3{\cdot}25$, $\therefore \frac{p \times r}{100} = £22{\cdot}25 \times 3{\cdot}25$.

$$\begin{array}{r} 22{\cdot}25 \\ 3{\cdot}25 \\ \hline 11125 \\ 4450 \\ 6675 \\ \hline 72{\cdot}3125 \end{array} \qquad \text{or} \qquad \begin{array}{l} 22{\cdot}25 \\ \phantom{22{\cdot}}3\frac{1}{4} \\ \hline 66{\cdot}75 \\ 5{\cdot}5625 \\ \hline 72{\cdot}3125 \end{array}$$

Ans. £72. 6*s.* 3*d.*

* Note that P is the *payment* to be made, and p the *principal.*

What is the premium on £415 at £2. 3*s*. 4*d*. % ?

	2·1667	or		$4·15 \times 2\frac{1}{6}$
	514			4·15
A	8 667		B	$2\frac{1}{6}$
	217			8·30
	108			·6916
	8·992			8·9916

Ans. £8. 19*s*. 10*d*.

The method A is universally applicable, but shorter methods, like B, will in certain cases suggest themselves to the computer.

What is the premium on £745 at £2. 8*s*. 7*d*. % ?

$$\begin{array}{r} 2·4291\dot{6} \\ 547 \\ \hline 17\ 004 \\ 972 \\ 121 \\ \hline 18·097 \end{array}$$

Ans. £18. 1*s*. $11\frac{1}{4}$*d*.

Find the total gain, if goods bought for £357. 12*s*. 6*d*. are sold at a profit of 9 %.

$$\begin{array}{r} 3·57625 \\ 9 \\ \hline 32·186 \end{array}$$

Ans. £32. 3*s*. $8\frac{3}{4}$*d*.

Exercise XLVII.

(1) Find the insurance on £560 at $2\frac{3}{4}$ %.

(2) Find the insurance on £712. 10*s*. 8*d*. at $3\frac{5}{8}$ %.

(3) Find $5\frac{3}{8}$ % on £77. 7*s*. 7*d*.

(4) Find the brokerage on 750 guineas at $1\frac{1}{4}$ %.

(5) Find the brokerage on 4593 doz. at 5*s*. $10\frac{1}{2}$*d*. per doz. at $\frac{7}{8}$ %.

(6) Find the commission on £2595 at $2\frac{1}{2}$ %.

(7) Find the premium of fire insurance on £1550 at 3*s*. 4*d*. %.

(8) Find the profit realized on £1470 at $12\frac{1}{2}$ %.

(9) Find the loss on goods bought for £370 and sold at a loss of $7\frac{3}{8}$ %.

(10) Goods bought for £275. 13*s*. 10*d*. are sold at a profit of $15\frac{7}{8}$ %. What were they sold for ?

(11) 17,500 dozen handkerchiefs were bought at 4*s.* $4\frac{1}{2}$*d.* per dozen, and sold at a profit of $17\frac{3}{4}$ %. How much were they sold for?

(12) A population of 357,600 increased $3\frac{1}{2}$ % in a certain year; the deaths were 11,920. Find the number of births.

§ 2. INTEREST.

Interest is money paid for the loan of money, and is calculated at so much % per annum. The calculation is rendered more complex than the questions in § 1 by the introduction of the element of Time. The difference between the two is that between Simple and Compound Proportion.

Find the interest on £340 at 4 % (per annum) for 3 years.

$$\begin{array}{llll} \text{If on £100} \uparrow & \text{for} \uparrow & \text{1 year we pay} & \text{£4,} \uparrow \\ \text{on £340} \downarrow & ,, \downarrow & \text{3 years} \quad ,, & \text{£}x. \downarrow \end{array}$$

$$\left.\begin{array}{r} 100 : 340 \\ 1 : 3 \end{array}\right\} = 4 : x.$$

$$3{\cdot}4 \times 12 = 40{\cdot}8. \qquad \textit{Ans. } \text{£40. 16}s.$$

Find the interest on £p at r % for t years.

$$\begin{array}{llll} \text{If on} \uparrow \text{£100 for} & \uparrow \text{1 year we pay} & \text{£}r, \uparrow \\ \text{on} \downarrow \text{£}p \quad ,, & \downarrow t \quad ,, & \text{£}x. \downarrow \end{array}$$

$$\left.\begin{array}{r} 100 : p \\ 1 : t \end{array}\right\} = r : x.$$

$$x = \frac{p \times r \times t}{100} = \text{Interest.} \qquad \textit{Ans. } I = \frac{p \times r \times t}{100}{}^{*}. \quad \text{(II.)}$$

Find the interest on £463. 14*s.* 9*d.* at $2\frac{3}{4}$ % for 1 year, 5 months.

Here $p = \text{£}463{\cdot}7375$, $r = 2{\cdot}75$, $t = 1\frac{5}{12} = 1{\cdot}41\dot{6}$,

$$\therefore \frac{p \times r \times t}{100} = \text{£}4{\cdot}637375 \times 2{\cdot}75 \times 1{\cdot}41\dot{6}.$$

$$\begin{array}{rl} 4{\cdot}637375 & \text{(4 places.)} \\ 572 & \\ \hline 9\ 2747 & \\ 3\ 2461 & \\ 2319 & \\ \hline 12{\cdot}7527 & \end{array}$$

* The money lent is called the *principal;* the yearly sum to be paid for £100, the *rate;* the interest + the principal = the *amount.* The letters p, r, t, I, have been chosen as the initials of the words principal, rate, time and interest.

12·7527 (3 places.)
766 141
12 753
5 101
128
76
7
1
18·066

Ans. £18. 1*s*. 4*d*.

or thus:

$2\frac{3}{4} \times 1\frac{5}{12} = \frac{11}{4} \times \frac{17}{12} = \frac{187}{48} = 3{\cdot}8958\dot{3}$.

4·637375 (3 places.)
85 983
13 912
3 710
417
23
3
18·065 *Ans.* £18. 1*s*. $3\frac{3}{4}$*d*.

Find the interest on £212. 10*s*. 4*d*. for $2\frac{3}{4}$ years at $2\frac{1}{2}$ % per annum.

$2\frac{3}{4} \times 2\frac{1}{2} = \frac{55}{8} = 6{\cdot}875$
$2{\cdot}1251\dot{6}$
5 786
12 750
1 700
148
11
14·610

or

$2{\cdot}1251\dot{6} \times 2{\cdot}5 \times 2{\cdot}75$.
52
4 2503
1 0626
5·3129
572
1 0626
3718
266
14·610 *Ans.* £14. 12*s*. $2\frac{1}{4}$*d*.

Experience alone will enable the computer to determine by inspection in which order it is best to take the factors.

Find the interest on £417. 7*s*. 9*d*. for 1 year, 10 months, at $4\frac{3}{8}$ %.

$4{\cdot}173875 \times 1{\cdot}8\dot{3} \times 4{\cdot}375$
5734
166955
12521
2921
209
18·2606
833 381
18 261
14 608
548
55
5
83·477

or

$4{\cdot}173875 \times \frac{35}{8} \times \frac{11}{6}$.

$\frac{35}{8} \times \frac{11}{6} = \frac{385}{48} = \frac{96{\cdot}25}{12} = 8{\cdot}0208\dot{3}$

4·1739
80 208
83391
83
3
83·477

Ans. £83. 9*s*. $6\frac{1}{2}$*d*.

Find the interest on £713. 10s. 9d. at $4\frac{5}{8}$ % for 79 days.

$$7\cdot135375 \times 4\cdot625 \times \tfrac{79}{365}.$$

```
  5)79
 73)15·8(·2̇16438'3'5'6'*        7·13537
     1 20                          5264
       470                      ---------
        320                     23 5415
         280                     4 2812
          61 &c.                   1427
                                    357
                                 --------
                                 33·0011
                                446 12
                                 --------
                                 6 600
                                   330
                                   198
                                    13
                                     1
                                 --------
                                 7·142
```

Ans. £7. 2s. $10\frac{1}{4}$d.

Exercise XLVIII.

I.

(1) Find the interest on £350 for 2 years at 5 %.
(2) ,, ,, £350 ,, 5 ,, 2 %.
(3) ,, ,, £765 ,, $1\frac{1}{2}$,, $2\frac{3}{4}$ %.
(4) ,, ,, £548. 16s. 3d. for 9 months at $4\frac{7}{8}$ %.
(5) ,, ,, £3456. 17s. 6d. for $4\frac{1}{2}$,, $3\frac{3}{4}$ %.
(6) ,, ,, £279. 12s. 10d. for 6 ,, $4\frac{3}{8}$ %.

II.

(1) Find the interest on £37. 18s. 9d. at $2\frac{7}{8}$ % for 47 days.
(2) ,, ,, £143. 10s. 8d. at $4\frac{5}{8}$ % for 100 days.
(3) ,, ,, £75. 16s. at $6\frac{1}{8}$ % for 42 days.
(4) ,, ,, 1000 guineas at 5 % for 12 days.
(5) ,, ,, £10490 at $4\frac{3}{4}$ % for 80 days.
(6) ,, ,, £876. 13s. 4d. at $3\frac{5}{8}$ % for 146 days.

* See Ch. VI. §§ 10 and 12; and note (*a*) that $61 + 12 = 73$; (*b*) that the accented figures are obtained at once by complementing 9. Or the period for $\frac{1}{73}$ might have been found at once from ring H, p. 163.

III.

(1) Find the interest on £1745. 12s. 8d. at £12. 15s. 8d. % for 130 days.

(2) „ „ £2495. 17s. 4d. at £13. 16s. 10d. % for 67 days.

(3) „ „ £65. 4s. 10d. at £5. 1s. 7$\frac{1}{4}$d. % for 91 days.

(4) „ „ £147. 16s. 6d. at £7. 12s. 10$\frac{1}{2}$d. % for 39 days.

(5) „ „ £439. 10s. 3d. at £9. 8s. 5d. % for 117 days.

(6) „ „ £455. 5s. 5d. at £8. 15s. 5$\frac{3}{4}$d. % for 19 days.

IV.

Find the amount on :

(1) £1483. 17s. 4d. for 1 year, 5 months, at 6$\frac{1}{4}$ %.
(2) £1517. 16s. 2$\frac{2}{5}$d. for 4 years, 8 months, at 5 %.
(3) £2045. 3s. 10d. for 76 days at 3$\frac{5}{8}$ %.
(4) £439. 11s. 5d. for 91 days at 4$\frac{2}{3}$ %.
(5) £254. 8s. 7d. for 145 days at £5. 8s. 11$\frac{1}{4}$d.%.
(6) £7777. 7s. 7d. for 77 days at £7. 7s. 7d.%.

V.

Find the interest for 1 minute on the national debt, £730,000,000, at 3 % per annum.

§ 3. Converse of Percentage.

We have seen (§ 1) that $P=\frac{p\times r}{100}$. Since $p\times r$ must be divided by 100 to obtain P, $100\times P=p\times r$. Again, since p must be multiplied by r to yield $100\times P$, p is the rth part of $100\times P$, or

$$p=\frac{100\times P}{r} \qquad \text{(III.)}$$

Similarly,

$$r=\frac{100\times P}{p} \qquad \text{(IV.)}$$

If, then, of the three quantities, P, p and r, any two be given, the third can be found by means of the formulæ (I.), (III.), (IV.).

Find the brokerage on £475 at $\frac{3}{4}$ %.

$$P=\frac{p\times r}{100}=\frac{475\times \cdot 75}{100}=4\cdot 75\times \cdot 75=3\cdot 5625.$$

Ans. £3. 11s. 3d.

On what principal will the brokerage at $\frac{3}{4}$ % amount to £3. 11*s.* 3*d.*?

$$p=\frac{100\times P}{r}=\frac{100\times 3\cdot5625}{\cdot75}=\frac{356\cdot25}{\cdot75}=475. \qquad \textit{Ans.}\ £475.$$

At what rate will the brokerage on £475 amount to £3. 11*s.* 3*d.*?

$$r=\frac{100\times P}{p}=\frac{100\times 3\cdot5625}{475}=\frac{356\cdot25}{475}=\cdot75. \qquad \textit{Ans.}\ \tfrac{3}{4}\ \%.$$

Exercise XLIX.

*(1) If goods are bought for £415 and sold for £500, what is the gain %?

(2) If the goods had been sold for £400, what would have been the loss %?

(3) On the breaking out of the war of 1870, the marine insurance of certain goods was raised from 2 to $5\frac{1}{2}$ %, making a difference of £140 in the premium to be paid. Find the value of the goods.

(4) A bought a horse for £40, and sold it to B at a profit of $8\frac{3}{4}$ %; B sold it at a loss of $7\frac{1}{2}$ % to C. What did C pay for the horse?

(5) What would C have paid if B had made $7\frac{1}{2}$ % profit?

(6) I bought French jewellery for fr. 7490, and paid an import duty of 15 % ad valorem; I sold it for £420. Find my gain or loss %, reckoning the £ at fr. 25·22.

(7) If $3\frac{3}{4}$ tons of sulphur are required to make 31 tons, 5 cwt. of gunpowder, what is the percentage of sulphur in gunpowder?

(8) At what rate will the brokerage on £1720. 16*s.* 8*d.* amount to £6. 9*s.* $0\frac{3}{4}$*d.*?

(9) On what sum will the brokerage at $\frac{5}{8}$ % amount to £10. 15*s.*?

(10) In a school of 80 children $17\frac{1}{2}$ % are girls. Find the number of boys.

(11) Find the rates to be paid on a house rented at 60 guineas and rated at 85 % of the rent, paying 1*s.* $3\frac{1}{2}$*d.* in the £.

(12) The total number of prisoners in a certain county was 493, of whom $\frac{17}{29}$ were males. The number of male prisoners in the following year was 272, and that of females $108\frac{1}{3}$ % of the former number. Find the total increase or decrease %.

* When not otherwise stated, find to 3 places.

(13) This year the number of out-door and in-door patients of a hospital were 1575 and 333 respectively; in the previous year the former were ·96 of this year's number, and the total was 1870. Find to one place the increase or decrease % of the in-door patients.

(14) A certain grocer's weekly business is as follows:

Goods.		Cost price.			Selling price.	
5 cwt. of raw sugar	@	36*s*.	per cwt.	@	4*d*.	per lb.
$2\frac{1}{2}$,, loaf sugar	,,	44*s*.	,,	,,	$5\frac{1}{2}$*d*.	,,
$\frac{1}{2}$,, coffee	,,	84*s*.	,,	,,	1*s*. 6*d*.	,,
which loses 20 % in roasting, and costs further 3*d*. per lb. duty and 2*s*. 6*d*. per cwt. roasting.						
40 lbs. of tea	,,	2*s*. 8*d*.	per lb.	,,	3*s*. 4*d*.	,,
$\frac{1}{2}$ cwt. of rice	,,	26*s*.	per cwt.	,,	4*d*.	,,
10 lbs. of cocoa	,,	1*s*. 1*d*.	per lb.	,,	1*s*. 4*d*.	,,
15 ,, ,,	,,	$6\frac{3}{4}$*d*.	,,	,,	8*d*.	,,
Sundries		£19. 12*s*.			£25. 8*s*. 6*d*.	

Find (*a*) the weekly profit realized; (*b*) the gain or loss % on each article; (*c*) the total gain %.

(15) If an investment of £91. 17*s*. 6*d*. yield me £3. 10*s*. per annum, what percentage do I get?

§ 4. Converse of Interest.

$$\text{Since}\quad I=\frac{p\times r\times t}{100},$$

$$100\times I=p\times r\times t,$$

$$\text{and}\quad \frac{100\times I}{p\times r}=t \qquad \text{(V.)}$$

$$\frac{100\times I}{p\times t}=r \qquad \text{(VI.)}$$

$$\frac{100\times I}{r\times t}=p \qquad \text{(VII.)}$$

If, then, of the four quantities, p, r, t, I, any three be given, the fourth can be found from the formulæ (II.), (V.), (VI.), (VII.).

Find the interest on £540 at 6 % for $4\frac{1}{2}$ months.

$$I=\frac{p\times r\times t}{100}=\frac{540\times 6\times \cdot 375}{100}=5\cdot 4\times 6\times \cdot 375=12\cdot 15. \qquad \textit{Ans.}\ £12.\ 3s.$$

In what time will £540 at 6 % yield an interest of £12. 3*s*.?

$$t=\frac{100\times I}{p\times r}=\frac{100\times 12\cdot 15}{540\times 6}=\frac{121\cdot 5}{324}=\cdot 375=\tfrac{375}{1000}\Big|\tfrac{3}{8}.$$

Ans. $\frac{3}{8}$ of a year, or $4\frac{1}{2}$ months.

At what rate will £540 in $4\frac{1}{2}$ months yield an interest of £12. 3*s.*?

$$r=\frac{100\times I}{p\times t}=\frac{100\times 12\cdot 15}{540\times\cdot 375}=\frac{1215}{54\times 3\cdot 75}=6. \qquad \textit{Ans.}\ 6\ \%.$$

What principal will in $4\frac{1}{2}$ months at 6 % yield an interest of £12. 3*s.*?

$$p=\frac{100\times I}{r\times t}=\frac{100\times 12\cdot 15}{6\times\cdot 375}=\frac{1215}{2\cdot 25}=540. \qquad \textit{Ans.}\ £540.$$

In what time will the interest on £318. 14*s.* 3*d.* at £4. 12*s.* % amount to £16. 4*s.* 9*d.*?

$$t=\frac{100\times I}{p\times r}=\frac{100\times 16\cdot 2375}{318\cdot 7125\times 4\cdot 6}=\frac{1623\cdot 75}{318\cdot 7125\times 4\cdot 6};$$ multiplying both terms by 8* $=\dfrac{12990}{2549\cdot 7\times 4\cdot 6}$.

$$\begin{array}{r}
25497\\
46\\
\hline
152982\\
101988\\
\hline
11\cancel{7}\cancel{2}8\cancel{6}\cancel{2})1299000(1\cdot 107\cancel{5}\cancel{5}\\
1261386\\
8852\\
642\\
56
\end{array}
\qquad
\begin{array}{r}
\left.\begin{array}{r}\cdot 1076\\ 563\end{array}\right\}\text{keeping one place.}\\
\hline
323\\
64\\
5\\
\hline
39\cdot\cancel{2}
\end{array}
\qquad \textit{Ans.}\ 1\text{ year, }39\text{ days.}$$

EXERCISE L.

(1) In what time will the interest on £455. 10*s.* amount to £1. 17*s.* $11\frac{1}{2}d.$ at 5 %?

(2) In what time will the interest on £368. 15*s.* 4*d.* amount to £15 at £6. 13*s.* 10*d.*%?

(3) In what time will a principal of £360 amount† to 400 guineas at 5 %?

(4) At what rate will the interest on £15 in $4\frac{1}{2}$ months amount to 7*s.* $10\frac{1}{2}d.$?

(5) What principal will, at £4. 18*s.* 9*d.*%, yield 500 guineas a year?

(6) In 2 years, 53 days, a principal of £1000 amounted to £1200. Find the rate charged.

* See foot-note, p. 188.

† See foot-note, p. 192.

(7) What sum will, at $8\frac{3}{4}$ %, yield 19 guineas interest in 3 months?

(8) Find x in each of the following:

	Principal.	Rate %.	Time.	Interest.
a.	£465,	$3\frac{1}{2}$,	47 days,	x.
b.	£x,	£4. 17*s.*,	1 yr., 4 mos.,	£70. 14*s.* 7*d.*
c.	£24,000,	x,	1 year,	£900.
d.	£230. 17*s.* 3*d.*,	$2\frac{1}{2}$,	x,	£3. 8*s.* 8*d.*

§ 5. Discount.

What sum of money will, at $3\frac{1}{2}$ %, in $2\frac{1}{2}$ months *amount to £75*?

The formula, $p=\frac{100\times I}{r\times t}$, will not avail us here, as I is unknown. We have recourse, therefore, to an artifice which is known in old books by the name of "Supposition," subsequently worn down to "Position," and which is applicable to a large class of questions besides the present.

Assume or *suppose* any sum of money whatever, say £86, laid out for the given time at the given rate, and find the interest thereon, and hence the amount.

$$\text{Interest on £86}=\frac{86\times3\cdot5\times2\cdot5}{100\times12}=\text{£}\cdot62708\dot{3}.$$

$$\text{Amount}\quad=\text{£}86\cdot62708\dot{3}.$$

We have now the following question in Proportion:

If £86 amounts to £86·62708$\dot{3}$,
£x " £75.

$$86\cdot62708\dot{3} : 75 = 86 : x.$$

```
86·627083) 64500000 (74·457
           3861042
            395959
             49451
              6137
```

Ans. £74. 9*s.* $1\frac{3}{4}$*d.*

Since for the "position" we may choose any sum we please, it is best to select the easiest, which obviously is in this case £100.

$$\text{Interest on £100}=\frac{\cancel{100}\times3\cdot5\times2\cdot5}{\cancel{100}\times12}=\cdot7291\dot{6}.$$

If, then, £100 amounts to £100·7291$\dot{6}$,
£x " £75.

$$100{\cdot}7291\dot{6} : 75 = 100 : x.$$

$$1{\cdot}0072916)75{\cdot}0000(74{\cdot}457$$
$$4\ 4896$$
$$4604$$
$$575$$
$$71$$

Ans. £74. 9*s.* 1¾*d.* as before.

Hence, generally:

What sum of money will, at $r\ \%$, in t years amount to £A?

Interest on £100 $= r \times t$,
Amount ,, $= 100 + r \times t$.

Hence the proportion:

$$100 + r \times t : A = 100 : x,$$

and
$$x = \frac{100 \times A}{100 + r \times t}.$$

The sum of money which at the given rate and time will amount to A is called the present value of A, which we will call V. Hence:

$$V = \frac{100 \times A}{100 + r \times t} \qquad \text{(VIII.)}$$

The sum of money paid in cash for the transfer of a debt due some time hence, should be the present value thus found. The difference between this present value and the amount of the debt is called Discount, which we will call D. Hence:

$$D = A - V \qquad \text{(IX.)}$$

D can, however, be obtained independently, thus:

If $r \times t$ is the discount on $100 + r \times t$,
x ,, A.

$$100 + r \times t : A = r \times t : x,$$

$$\therefore\ D = \frac{A \times r \times t}{100 + r \times t} \qquad \text{(X.)}$$

Discount questions occur most frequently in connection with Bills of Exchange.

Find the *true* discount on a bill for £387. 14*s.* 10*d.*, dated May 5th, at 3 months, and discounted June 4th, interest being calculated at 4 %.

This means that £387. 14*s.* 10*d.* will fall due on August 5th, but in most countries the usage of commerce allows 3 days' "grace," and accordingly the money will be payable August 8th. From June

4th to August 8th are 65 days, which is the time the bill has yet to run.

$$\therefore\ D=\frac{£387.\ 14s.\ 10d. \times 4 \times \frac{65}{365}}{100+4\times\frac{65}{365}}.$$

$$r\times t=4\times\frac{\overset{13}{\cancel{65}}}{\underset{73}{\cancel{365}}}=\frac{52}{73}=\cdot\dot{7}123287\dot{6}$$

73) 520 (·$\dot{7}$1232876
90
170
240
21
&c.

$$D=\frac{387\cdot741\dot{6}\times\cdot\dot{7}123287\dot{6}}{100\cdot\dot{7}123287\dot{6}}$$

387·742
9232 17
―――――
271419
3877
775
116
8
3
―――――
100·7123) 276·198 (2·742
74 773
4 275
247

Or see ring G, p. 163.

Ans. £2. 14*s.* 10¼*d.*

Practically, in England, the true discount is never calculated, but the banker deducts simple interest from the amount of the bill. Comparing the formulæ for I and D, we find the numerators the same, while the denominator in the former is 100, and in the latter $100+r\times t$.

In the above sum the interest is:

$$\frac{387\cdot741\dot{6}\times\cdot\dot{7}123287\dot{6}}{100}=2\cdot76198=£2.\ 15s.\ 2\tfrac{3}{4}d.$$

Deducting the true discount we get $4\frac{1}{2}d.$, which is the banker's surcharge.

Exercise LI.

(1) Find the present value of:

a. £427. 10*s.* 6*d.*,	discounted at	$3\frac{1}{2}$ %,	due in	3	months.
b. £6359. 18*s.* 4*d.*	"	$4\frac{3}{4}$ %,	"	4	"
c. £794. 11*s.*	"	5 %,	"	6	"
d. £82. 13*s.* 4*d.*	"	$5\frac{3}{8}$ %,	"	70 days.	

e. £445. 10*s.*, discounted at $3\frac{7}{8}$ %, due in 94 days.

f. £1250 „ £2. 17*s.* 8*d.* %, due in 114 days.

(2) Find the discount on a bill of:

a. £700, due March 5—8, discounted January 4, at 5 %.

b. £376, due August 20—23, discounted May 11, at $4\frac{1}{2}$ %.

c. £40. 10*s.* 6*d.*, due September 30—October 3, discounted September 17, at $3\frac{1}{4}$ %.

d. £461. 3*s.* 6*d.*, due October 9—12, discounted May 5, at $7\frac{1}{8}$ %.

(3) Find the present value of a bill of:

a. £629. 12*s.*, drawn May 11, at 3 months, discounted June 2, at $5\frac{3}{8}$ %.

b. £485. 19*s.* 3*d.*, drawn July 31,* at 4 months, discounted November 11, at $8\frac{1}{2}$ %.

c. £374. 16*s.*, drawn March 10, at 90 days, discounted April 1, at $6\frac{7}{8}$ %.

(4) Find the difference between simple interest and discount on:

a. £760 at $3\frac{5}{8}$ % for 68 days.

b. £1848. 10*s.* 10*d.* at $4\frac{2}{5}$ % for 80 days.

c. £2466. 13*s.* 4*d.* at $5\frac{3}{5}$ % for 99 days.

In a similar manner may be treated questions such as the following:

For how much should goods worth £540 be insured at 3 %, so as to recover in case of loss both goods and premium?

Goods worth £97 ↓ should be insured for ↓ £100.
„ £540 ↓ „ „ ↓ £*x*.

$$97 : 540 = 100 : x.$$

```
97) 54000 (556·701
      550
       650
        680
         100
```

Ans.* £556. 14*s.

If goods sold at a profit of $12\frac{1}{2}$ % realized £360, find the cost price.

We have only to remember that gain or loss is always calculated on the cost price.

* This will be due November 30—December 3.

Cost price, £100, ↑ Selling price, £112·5. ↑
£x ↓ £360. ↓

$$112{\cdot}5 : \cancel{360} = 100 : x$$
$$\cancel{450} \quad 4 \quad \cancel{400}$$
$$\cancel{5} \qquad 80$$
$$4 \times 80 = 320$$

Ans. £320.

Exercise LII.

(1) If by selling an article at £1. 1*s.* 9*d.* a pound I gain 16 %, what was the prime cost?

(2) I bought 100 gallons of brandy at 17*s.* 6*d.* per gallon; $9\frac{1}{2}$ gallons are lost by leakage; the remainder is put into bottles holding $1\frac{1}{3}$ pints; at what price per bottle must it be sold so as to gain $18\frac{1}{2}$%?

(3) For how much should goods worth £635 be insured at $2\frac{1}{2}$ % to recover the value of goods and premium?

(4) Also goods worth £100 at 5 %?

(5) „ £67. 10*s.* at 4 %?

(6) Goods bought at £2. 5*s.* 10*d.* per cwt. are sold at £2. 14*s.* 1*d.*; what is the gain per cent.?

(7) Of goods worth £1000, one-third is sold at a profit of 15 %; for how much must the remainder be sold to gain 20 % on the whole?

(8) I bought sugar at £1. 18*s.* 6*d.* per cwt.; at how much per lb. must I sell it to lose 4 %? (To be worked accurately.)

(9) If by selling wine at 15*s.* a gallon I lose 10 %, at what price must I sell it to gain 15 %?

(10) A person buys coffee at £5. 12*s.* 6*d.* per cwt, and chicory at £2. 5*s.* 5*d.*, and mixes them in the proportion of 2 of chicory to 5 of coffee. He retails the mixture at 1*s.* 3*d.* per lb.; what is his gain or loss %?

(11) I bought 4 cwt. of sugar at 4*d.* per lb., and 7 cwt. at 5*d.* per lb.; at how much per lb. must I sell the mixture to gain 20 %?

(12) A bought 150 eggs at 2 a penny, and 150 at 3 a penny; he sold them all at 5 for 2*d.* How much % did he gain or lose?

(13) I bought 580 metres of silk at 6·65 fr. per metre, and sold 300 yards at 6*s.* $9\frac{1}{2}$*d.* per yard, and the remainder at 7*s.* per yard;

find the gain or loss %, reckoning 1 metre = 39·37 in., and £1 = 25·22 fr.

(14) If apples are bought at 4 for $1\frac{1}{2}d.$, how many should be sold for $3\frac{1}{2}d.$ to gain 75 %?

(15) I bought a Geneva watch and paid duty at the rate of 10 %. I sold the watch for £21, making a profit of 15 % on my whole outlay. Find the original cost in francs at $9\frac{1}{2}d.$ each.

§ 6. Compound Interest.

When the interest is not drawn, but is added from time to time, as it becomes due, to the principal, and is calculated for each interval on the amount thus increased, the total charge made for the loan is said to be reckoned at Compound Interest.

Find the compound interest on £256 at 4 % for $1\frac{1}{2}$ years, reckoned half-yearly.

In this case $\frac{r \times t}{100}$ will for each half-year be $\frac{4 \times \cdot 5}{100} = \cdot 02$; hence each successive amount will have to be multiplied by ·02.

1st method.

Mod. op.:

Original principal..................................	£256
Interest for 1st period, £256 × ·02..............	5·12
Amount due at end of 1st period	261·12
Interest for 2nd period, £261·12 × ·02	5·2224
Amount due at end of 2nd period	266·3424
Interest for 3rd period, £261·3424 × ·02	5·3268...
Amount due at end of 3rd period	271·6692
Deduct original principal........................ ..	256
Compound interest	£15·6692

Ans. £15. 13s. $4\frac{1}{4}d.$

Multiplying any number by ·02 and adding the product to the original number, is equivalent to multiplying at once by 1·02, which can be done in one line (Part I. p. 99), thus:

256
261·12
266·3424
271·6692...
256

15·6692 *Ans.* £15. 13*s.* 4⅜*d.*

2nd method. Find the amount at compound interest of £845. 12*s.* 8*d.* at 5 % for 5 years, reckoned yearly.

First find the amount of £1 at compound interest at 5 % for 5 years. Uniform multiplier $1 + \frac{r \times t}{100} = 1·05$.

Original principal	....	£1·
Amount at end of 1st year	....	1·05
,, 2nd ,,	....	1·1025
,, 3rd ,,	....	1·157625
,, 4th ,,	....	1·21550625
,, 5th ,,	....	1·2762815625

Now multiply the last line by the principal, 845·63̇ :

1·276281 × 845·63̇
83 36548

10 21025
51051
6381
766
38
4

1079·265 *Ans.* £1079. 5*s.* 3⅝*d.*

A computer who has often to calculate compound interest would find it useful to make tables similar to the above, carried out further according to need, for various rates and periods.

EXERCISE LIII.

(1) Find the compound interest on :

a. £585 for 2 years, reckoned quarterly, at 4 % per annum.
b. £1000 for 10 years at 5 %, reckoned yearly.
c. £60 at 5 % for 1 year, reckoned monthly.*
d. £145. 17*s.* 6*d.* for $2\frac{1}{2}$ years at $3\frac{1}{2}$ %, reckoned half-yearly.
e. £624. 12*s.* 8*d.* for 5 years at $3\frac{7}{8}$ %, reckoned yearly.
f. £815. 13*s.* 9*d.* at £2. 17*s.* 9*d.* %, for 3 years, reckoned half-yearly.

(2) Find the difference between simple and compound interest reckoned half-yearly, on £850 for 3 years at $5\frac{1}{2}$ % per annum.

(3) Find the difference between simple and compound interest reckoned yearly, on £738. 15*s.* at $4\frac{3}{4}$ % per annum, for 4 years.

(4) Find the difference in compound interest between reckoning yearly, half-yearly and quarterly, on £1000 for 3 years at 4 % per annum.

§ 7. CONVERSE OF COMPOUND INTEREST.

What sum of money will in 3 years amount to £3000 at $3\frac{1}{2}$ % per annum compound interest, reckoned half-yearly?

Uniform multiplier $1 + \frac{r \times t}{100} = 1{\cdot}0175$.

Table for £1 (keeping 8 places).

Original principal			......	£1·
Amount at end of 1st period			......	1·0175
„	2nd	„	$(1{\cdot}0175)^2$......	1·03530625
„	3rd	„	$(1{\cdot}0175)^3$......	1·05342410...
„	4th	„	$(1{\cdot}0175)^4$......	1·07185903...
„	5th	„	$(1{\cdot}0175)^5$......	1·09061656...
„	6th	„	$(1{\cdot}0175)^6$......	1·10970235...

* This question can by the second method be worked thus :
(Amount of £1 for 1 month $)^2$ = amount for 2 months
(Amount „ 2 months$)^2$ = „ 4 „
(Amount „ 4 months$)^3$ = „ 12 „ (Cf. Ch. XI. § 25.)

If £1 amounts to £1·10970235,
x „ £3000.

$$1·10970235 : 3000 = 1 : x.$$

$$\begin{array}{l} 110970235)\,30000000\,(2703·427 \\ \quad\quad\quad\quad\quad\;\; 7805953 \\ \quad\quad\quad\quad\quad\quad 38037 \\ \quad\quad\quad\quad\quad\quad\; 4746 \\ \quad\quad\quad\quad\quad\quad\;\; 307 \\ \quad\quad\quad\quad\quad\quad\;\; 85 \end{array}$$

Ans. £2703. 8*s.* 6½*d.*

Exercise LIV.

What sum of money will amount to:

a. £500 at 5 % per annum compound interest, reckoned yearly, in 4 years?

b. £750 at 4 %, reckoned six-monthly, in 2½ years?

c. 1000 guineas at 3½ %, in 5 years, reckoned yearly?

§ 8. Equation of Payments.

Occasionally it is proposed to find the time at which several debts due at different times might be discharged by a single payment of their sum total.

Find the "equated time" for the following bills:

£800 due in 4 months at 5 %.
£560 „ 6 „
£375 „ 9 „

$$\text{Present value of the } £800 = \frac{800 \times 100}{100 + 5 \times \dot{3}} = 786·8852$$

$$\text{„ „ } £560 = \frac{560 \times 100}{100 + 5 \times ·5} = 546·3414$$

$$\text{„ „ } £375 = \frac{375 \times 100}{100 + 5 \times ·75} = 361·4458$$

$$\text{„ „ } £1735 = \quad 1694·6724$$

In what time, then, will £1694·6724 amount to £1735 at 5 %? (Formula V. p. 197.)

$$I = 1735 - 1694·6724 = 40·3276$$

$$t = \frac{100 \times 40·3276}{1694·6724 \times 5} = ·476 \text{ of a year} = 173 \text{ days.}$$

Ans. 173 days.

This is reckoned rigorously, by discount; but in commerce, when such questions occur, a mere average is struck:

$$\frac{800 \times 4 + 560 \times 6 + 375 \times 9}{1735} = 5{\cdot}72 \text{ months (about 171 days).}$$

§ 9. STOCKS AND INVESTMENTS.

Stock is another word for capital, not expressly divided into shares, in any joint undertaking, whether that undertaking be the loan of money to a government or the placing it in the hands of the managers of a trading company.

From various causes the market value of this stock fluctuates. When the market price is the same as the sums for which the holder stands credited in the books of the government, the stock or shares are said to be at par; when less, at a discount; when more, at a premium. We must therefore carefully distinguish between two kinds of principal, viz., *stock* money or *sterling* value, the former being the original price, or at any rate the sum credited to the holder's name in the books of the company or government; the latter, the market value per cent.

On June 22, 1870, Consols (i.e. stock in the Consolidated Three per Cent. Annuities of England) are quoted at $92\frac{5}{8}$, which means that a claim on the government for the yearly interest on £100 can be bought in the market for £92. 12*s.* 6*d.*, or that £100 stock = £92. 12*s.* 6*d.* sterling.

This leads to such questions as the following:

a. How much stock for a given sum in sterling money?

b. How much sterling money for a given quantity of stock?

How much stock can be bought for £2416. 10*s.* at $92\frac{5}{8}$?*

x stock	£2416·5 sterling
$92\frac{5}{8}$ sterling	100 stock

```
92625) 241650000 (2608·906
          564000
           825000
            84000
              637
               81
```

Ans. £2608. 18*s.* $1\frac{1}{4}$*d.*

* All problems worked by chain rule can also be worked by direct proportion, and *vice versa* (see p. 89).

How much money will be realized by selling £815 Consols at $92\frac{5}{8}$?

$$\begin{array}{r|l} x \text{ sterling} & 815 \text{ stock} \\ 100 \text{ stock} & 92\frac{5}{8} \text{ sterling} \end{array}$$

$$\begin{array}{r} \cdot 92625 \\ 518 \\ \hline 741000 \\ 9263 \\ 4631 \\ \hline 754\cdot 894 \end{array}$$

Ans. £754. 17*s.* $10\frac{1}{2}$*d.*

Exercise LV.

(1) How much stock can be bought for £8450 at $93\frac{1}{4}$?

(2) " " £748. 16*s.* at $89\frac{7}{8}$?

(3) What will be realized by selling out £350 Consols at $91\frac{1}{2}$?

(4) " " £68. 4*s.* 6*d.* Consols at $87\frac{3}{4}$?

§ 10. As a matter of practice, every transfer of shares is effected by a broker, who charges $\frac{1}{8}$ % to both buyer and seller. The purchaser has thus to pay $\frac{1}{8}$ % more, and the vendor receives $\frac{1}{8}$ % less, than the price quoted.

How much stock can be bought for £520 at $93\frac{1}{2}$, with $\frac{1}{8}$ brokerage?

Each £100 stock costs $93\frac{1}{2} + \frac{1}{8} = £93\frac{5}{8}$ sterling.

$$\begin{array}{r|l} x & 520 \\ 93\frac{5}{8} & 100 \\ 749 & 8 \end{array}$$

$$\begin{array}{l} 749)\ 416000\ (555\cdot 407 \\ \quad\quad 4150 \\ \quad\quad 4050 \\ \quad\quad\ 3050 \\ \quad\quad\quad 54 \end{array}$$

Ans. £555. 8*s.* $1\frac{3}{4}$*d.*

How much will be realized by selling out £520 stock at $89\frac{5}{8}$; brokerage $\frac{1}{8}$ %?

Each £100 stock realizes $89\frac{5}{8} - \frac{1}{8} = £89\frac{1}{2}$.

$$\begin{array}{r|l} x & 520 \\ 100 & 89\cdot 5 \end{array}$$

$8\cdot 95 \times 52 = 465\cdot 4.$ *Ans.* £465. 8*s.*

EXERCISE LVI.

Find the sum to be paid for:

(1) £8350 3 per cent. Consols, at $94\frac{1}{2}$, brokerage $\frac{1}{8}$.
(2) £746. 18*s.* 6*d.* 3 per cent. Consols, at $91\frac{7}{8}$, brokerage $\frac{1}{8}$.
(3) £844. 4*s.* 4*d.* 3 per cent. Consols, at $89\frac{1}{4}$, brokerage $\frac{1}{8}$.
(4) £768. 17*s.* 11*d.* 3 per cent. Consols, at 88, brokerage $\frac{1}{8}$.

What will be realized by selling out:

(5) £600 3 per cent. Consols, at 90, brokerage $\frac{1}{8}$.
(6) £937. 12*s.* 11*d.* 3 per cent. Consols, at $90\frac{1}{2}$, brokerage $\frac{1}{8}$.
(7) £666. 13*s.* 4*d.* 3 per cent. Consols, at $86\frac{5}{8}$, brokerage $\frac{1}{8}$.
(8) £27468. 10*s.* 3 per cent. Consols, at $91\frac{1}{4}$, brokerage $\frac{1}{8}$.

§ 11. The operations of most frequent occurrence on the stock exchange lead to questions like the following:

Bought £16,000 Consols at $93\frac{5}{8}$, and sold out at $94\frac{1}{8}$. Find profit, allowing for brokerage $\frac{1}{8}$ % on each operation.

$$\text{Profit on £100 stock} = (94\tfrac{1}{8} - \tfrac{1}{8}) - (93\tfrac{5}{8} + \tfrac{1}{8}) = 94 - 93\tfrac{3}{4} = \tfrac{1}{4},$$
$$\text{,, \quad £16,000} \quad = 160 \times \tfrac{1}{4} = \text{£}40.$$

Bought fr. 25,000 in the French Rentes at 71·75, and sold at 66·90. Find the loss (brokerage $\frac{1}{8}$ % both ways).

$$\text{Loss on 100 fr. stock} = (71{\cdot}75 + {\cdot}125) - (66{\cdot}90 - {\cdot}125) = 5{\cdot}1,$$
$$\text{,, \quad fr. } 25{,}000 = 250 \times 5{\cdot}1 = 1275 \text{ fr.}$$

EXERCISE LVII.

(1) If the English 3 per cent. funded debt amounts to £723,120,000, find the diminution of its value in the market caused by the outbreak of the German-French war in 1870. Quoted $92\frac{5}{8}$ on June 22, and $88\frac{3}{8}$ August 5.

(2) Find the profit or loss on each of the following operations (brokerage $\frac{1}{8}$ % on each transaction):

	Stock.	Bought in at.	Sold out at.
a.	£80,000,	$91\frac{5}{8}$,	90.
b.	£17,500,	$88\frac{7}{8}$,	$89\frac{1}{8}$.
c.	£51,600,	$86\frac{1}{4}$,	$89\frac{1}{2}$.
d.	fr. 30,000,	65·75,	67·35.

§ 12. What yearly income will be derived from the investment of £7850 in the 3 per cents. at 90?

£90 sterling buys an annuity of £3.

$$\begin{array}{c|l} x & 785\cancel{0} \\ \cancel{90} & \cancel{3} \\ 3 & \end{array} \qquad \begin{array}{r} 3)\underline{785 \quad 0 \quad 0} \\ £261.\ 13s.\ 4d. \end{array}$$

Ans. £261. 13*s.* 4*d.*

How much must I invest in the 3 per cents. at $90\frac{5}{8}$ to obtain an income of £150 a year (brokerage $\frac{1}{8}$ %)?

Each £100 share costs $90\frac{5}{8}+\frac{1}{8}=90\frac{3}{4}$.

$$\begin{array}{c|l} x & £\cancel{150} \\ £\cancel{3} & 90{\cdot}75 \\ & 50 \end{array} \qquad 90{\cdot}75\times 50=4537{\cdot}5$$

Ans. £4537. 10*s.*

If I invest in the 4 per cents. at $89\frac{1}{2}$, what rate of interest do I get for my money?

$$\begin{array}{c|l} x & 100 \\ 89{\cdot}5 & 4 \end{array} \qquad \begin{array}{r} 8\cancel{9}\cancel{5})\,4000\,(4{\cdot}469 \\ 4200 \\ 620 \\ 83 \\ 2 \end{array}$$

Ans. £4. 9*s.* $4\frac{3}{4}$*d.* %.

Exercise LVIII.

(1) What yearly income will be derived from £895 invested in the 5 per cents. at 105?

(2) What yearly income is derived from the investment of £10,000 in the 3 per cent. Consols at $91\frac{1}{2}$?

(3) Find the yearly income derived from investing £750 in the 5 per cent. Austrian Metalliques at $48\frac{7}{8}$.

(4) How much must I invest in the 4 per cents. at $93\frac{5}{8}$ to realize a yearly income of £120?

(5) How much must be invested in the $3\frac{1}{2}$ per cents. at $87\frac{3}{4}$ to realize a six-monthly dividend of £75?

(6) Work the above five sums, allowing $\frac{1}{8}$ % brokerage.

(7) Find the yearly rate of interest obtained from the following investments:

a. Consols 3 per cent. at $88\frac{5}{8}$.

b. French Rentes 3 per cent. at 66·90.

c. Prussian 5 per cent. at 88.

d. Russian $4\frac{1}{2}$ per cent. at 86.
e. Turkish 6 per cent. (1854) at $81\frac{1}{2}$.
f. Turkish 5 per cent. (1865) at $38\frac{1}{2}$.

§ 13. Which is the most profitable stock for investment, the 4 % at 85, or the 3 % at 63?

Suppose £100 invested in each kind, the 4 % will yield $\frac{100 \times 4}{85} = 4{\cdot}706$; the 3 % will yield $\frac{100 \times 3}{63} = 4{\cdot}762$. *Ans.* The 3 %.

It is simpler to suppose £1 invested in each kind, and then the resulting fractions are $\frac{4}{85}$ and $\frac{3}{63}$, which decimalized become ·04706 and ·04762. $\frac{3}{63}$, then, is the larger fraction, and the 3 % are the better investment. *Ans.* The 3 %.

§ 14. Find the difference in income between investing £866. 13*s.* 4*d.* in the 5 per cents. at $107\frac{5}{8}$ and in the $4\frac{1}{2}$ per cents. at $95\frac{1}{4}$, brokerage $\frac{1}{8}$.

£1 in the 5 per cents. yields $\frac{5}{107\frac{3}{4}} = £{\cdot}0464037$

£1 „ $4\frac{1}{2}$ „ $\frac{4\frac{1}{2}}{95\frac{3}{8}} = £{\cdot}0471821$

Difference in income on £1 $= {\cdot}0007784$

„ „ £866·$\dot{6}$ $= 866{\cdot}\dot{6} \times {\cdot}0007784$.

$$\begin{array}{r} {\cdot}0007784 \\ 768 \\ \hline 622 \\ 46 \\ 5 \\ \hline {\cdot}673 \end{array}$$

Ans. 13*s.* $5\frac{3}{4}d.$

EXERCISE LIX.

(1) Which of the following pairs is the more profitable investment:

a. 4 per cents. at 95, and $3\frac{1}{2}$ per cents. at 84.
b. $3\frac{1}{2}$ „ $67\frac{1}{4}$, and 4 „ $81\frac{1}{2}$.
c. $5\frac{3}{4}$ „ $110\frac{1}{2}$, and 3 „ $67\frac{5}{8}$.

(2) Find the differences in income, supposing £584. 10*s.* were invested in each of the pairs in (1).

(3) Find the difference in income between investing £1500 in the 3 per cent. Consols at $88\frac{7}{8}$, and in $12\frac{1}{2}$ per cent. Bank stock at $185\frac{1}{2}$, brokerage $\frac{1}{8}$.

§ 15. What should be the price of the 3 per cents. in order that investments in them should make 5 % interest?

$$\begin{array}{c|l} x & 3 \\ \cancel{5} & \cancel{100} \\ & 20 \end{array} \qquad 3 \times 20 = 60. \qquad \textit{Ans. } 60.$$

What should be the price of the $3\frac{1}{2}$ per cents. to yield the same rate of interest as the 4 per cents. at 95?

$$\begin{array}{c|l} x & 3{\cdot}5 \\ 4 & 95 \end{array} \qquad \frac{3{\cdot}5 \times 95}{4} = 83\tfrac{1}{8}. \qquad \textit{Ans. } 83\tfrac{1}{8}$$

EXERCISE LX.

(1) Find the price of $4\frac{1}{2}$ % to equal the $3\frac{1}{2}$ % at $88\frac{1}{2}$.
(2) „ $3\frac{3}{4}$ „ 4 „ $92\frac{1}{4}$.
(3) „ 4 „ 5 „ $100\frac{7}{8}$.
(4) „ $2\frac{1}{2}$ „ $3\frac{1}{2}$ „ $68\frac{5}{8}$.
(5) What should the 5 % be when the 3 % are at $89\frac{1}{2}$?
(6) „ $3\frac{1}{2}$ „ $5\frac{3}{4}$ „ par?

§ 16. If I transfer £3850 from the $3\frac{1}{2}$ per cents. at $91\frac{3}{8}$ to the 5 per cents at $102\frac{1}{2}$, find the alteration in (*a*) the amount of stock held, (*b*) the yearly income, (*c*) the rate of interest for my money.

a.

$$\begin{array}{r|l} x^{*} & 3850 \\ 1\cancel{00} & 91{\cdot}375 \\ 102{\cdot}5 & 1\cancel{00} \end{array} \qquad \frac{3850 \times 91{\cdot}375}{102{\cdot}5} = 3432{\cdot}134.$$

$$\begin{array}{lr} & 3850 \\ & 3432{\cdot}134 \\ \hline \text{Difference} & 417{\cdot}866 \end{array} \qquad \textit{Ans. } £417.\ 17s.\ 4d. \text{ decrease.}$$

b.

$$\begin{array}{r|l} x & 3850 \\ 100 & 91{\cdot}375 \\ 102{\cdot}5 & 5 \end{array} \qquad\qquad \begin{array}{r|l} x & 3850 \\ 100 & 3{\cdot}5 \end{array}$$

Ans. 171·607. *Ans.* 134·75.

$$\begin{array}{lrl} & 171{\cdot}607 & \text{income in new investment.} \\ & 134{\cdot}75 & \quad „ \quad \text{old} \quad „ \\ \hline \text{Difference} & 36{\cdot}857 & \end{array} \qquad \textit{Ans. } £36.\ 17s.\ 1\tfrac{3}{4}d. \text{ increase.}$$

* Read, How much 5 per cent. stock for £3850 $3\frac{1}{2}$ per cent. stock, if £100 $3\frac{1}{2}$ per cent. stock yields £91·375 sterling, and £102·5 sterling buys £100 5 per cent. stock?

c.

$$\begin{array}{r|l} x & 100 \\ 91\frac{3}{8} & 3\frac{1}{2} \end{array} \qquad\qquad \begin{array}{r|l} x & 100 \\ 102{\cdot}5 & 5 \end{array}$$

Ans. 3·830. *Ans.* 4·878.

4·878 rate of interest in new investment.
3·830 ,, old ,,
Difference 1·048 *Ans.* £1. 0*s.* 11½*d.* % increase.

Exercise LXI.

Find the alteration in the amount of stock held, the income derived and the rate %, if the following transfers are effected:

	Amount transferred	from	at	to	at.
(1)	£7850,	$3\frac{1}{2}$ %,	66·90,	5 %,	90.
(2)	£5465,	$4\frac{3}{4}$,,	$90\frac{1}{2}$,	$2\frac{1}{2}$,,	$47\frac{1}{2}$,
(3)	£690,	3 ,,	$76\frac{1}{2}$,	4 ,,	$82\frac{5}{8}$,
(4)	£3567. 14*s.* 10*d.*,	3 ,,	88,	$3\frac{1}{2}$,,	par.
(5)	£1487. 11*s.* 8*d.*,	$3\frac{1}{4}$,,	90,	$6\frac{1}{2}$,,	180.
(6)	10,000 guineas,	3 % Consols,	$88\frac{3}{8}$,	5 % Italians	at $46\frac{1}{2}$.

§ 17. Shares. Calculations on stock are, with slight alterations, applicable also to shares, and it is most convenient preliminarily to suppose share capital converted into stock, as this enables us to calculate by percentages, irrespective of the issue value of the shares. Thus shares issued at £20, "fully paid up," and sold at a discount of £6, i.e. for £14, may be compared to £100 stock at £70, share and price being both multiplied by 5.

If shares in a certain company, issued at 25 and bought at 2 premium, yield a six-monthly dividend of £1. 16*s.*, find (*a*) the sterling value of the shares per cent., (*b*) the rate of interest, and (*c*) the yearly income derived from investing £540 in these shares.

a. $\begin{array}{r|l} x & 100 \\ 25 & 27 \end{array}$ *Ans.* £108.

b. $\begin{array}{r|l} x & 100 \\ 27 & 2\times1{\cdot}8 \end{array}$ *Ans.* £13⅓ %.

c. $\begin{array}{r|l} x & 540 \\ 27 & 2\times1{\cdot}8 \end{array}$ *Ans.* £72.

The issue price of shares is usually not "paid up" on application, but "called in" by the managers according to necessity or stipulation.

Find the rate of interest on an investment in railway shares issued at £50, on which were paid £10 on application and four successive

calls of £5 each, bought at $1\frac{5}{8}$ premium, and yielding a six-monthly dividend of 17*s*. 6*d*. per share.

Capital invested: $£10 + 4 \times £5 + £1\frac{5}{8} = £31.\ 12s.\ 6d.$

$$\begin{array}{r|l} x & 100 \\ 31\cdot625 & \cdot875 \times 2 \end{array} \qquad \begin{array}{l} 253)\,1400\,(5\cdot533 \\ \quad\;\; 1350 \\ \quad\;\;\; 850 \\ \quad\;\;\;\; 91 \end{array}$$

Ans. £5. 10*s*. 8*d*.%.

EXERCISE LXII.

(1) Find the six-monthly dividend derived from investing 1000 guineas in £50 railway shares at par, yielding $3\frac{1}{2}$ per cent. per annum.

(2) Find the price per cent. of mining shares issued at £15 and sold at $2\frac{1}{2}$ discount.

(3) Find the rate of interest on shares, £35 paid up, yielding a half-yearly dividend of £2. 14*s*.

(4) Find the rates of interest obtained by investing in the following railway shares:

a. London and North Western,	quoted at $124\frac{1}{2}$,	paying 6 %	per an.
b. Midland,	" 124,	" $6\frac{1}{4}$ %	"
c. Great Western,	" $67\frac{3}{4}$,	" $3\frac{9}{40}$ %	"
d. South Eastern,	" 67,	" $2\frac{1}{2}$ %	"

(5) Find the yearly income derived from the investment of every £100 in the Austrian 5 per cents. at 48·55, deducting 16 % income tax.

(6) I invested £1450 in the 3 per cents. at $88\frac{1}{2}$, and sold out when they had risen $2\frac{1}{4}$ %. What was my gain?

(7) How much stock must I sell out of the $3\frac{1}{2}$ per cents. at $81\frac{7}{8}$ to enable me to buy £5000 4 per cent. stock at $94\frac{1}{2}$, brokerage $\frac{1}{8}$ in each transaction?

(8) What is the price of stock if £8000 stock can be bought for £5830?

(9) My half-yearly dividend from the 3 per cents. is £247. 10*s*. How much stock do I hold, and what shall I realize by selling out at $89\frac{1}{2}$, brokerage $\frac{1}{8}$?

(10) How much stock must I sell out from the 4 per cents. at $96\frac{7}{8}$ to raise a sufficient sum for discounting a bill for £1000, due 52 days hence and discounted at $5\frac{1}{2}$ %?

(11) Find the yearly income derived from investing a legacy of £4583. 10*s*. in the 3 per cent. Consols at $91\frac{7}{8}$, allowing for legacy duty 5 %, brokerage $\frac{1}{8}$ %, deducting an income tax of 5*d*. in the £.

(12) I invested £1460 in the $4\frac{1}{2}$ per cents. at $100\frac{1}{4}$, and sold when they had fallen, losing £100, inclusive of the double brokerage. Find the selling price.

(13) What would this selling price have been if I had cleared £100?

(14) I bought Jan. 1st, £5000 Consols at $92\frac{3}{8}$; I sold Feb. 10th, £1500 of this stock at $93\frac{7}{8}$, March 12th, £2000 at 93, and the remainder April 1, at $92\frac{5}{8}$, brokerage $\frac{1}{8}$ % on each transfer. Find my total gain or loss, supposing that I could have made 5 % per annum by other investments.

(15) I invested £2460 in the 3 per cents. at $90\frac{3}{8}$, with $\frac{1}{8}$ % brokerage; on their falling to $87\frac{1}{2}$, I sold out, paying again $\frac{1}{8}$ % brokerage, and put my money out on a mortgage at $4\frac{1}{2}$ %. Find the alteration in my capital and in my yearly income.

(16) I bought £5000 stock at $88\frac{1}{2}$. At what price must I sell it to gain £100?

(17) If I buy at 85, at what price must I sell to make $12\frac{1}{4}$ % profit, brokerage $\frac{1}{8}$ on each transaction.

(18) I invested money in the 3 per cents. at $75\frac{3}{8}$, and after drawing half a year's dividend, I sold out at a rise of $1\frac{7}{8}$ %, increasing my capital altogether by £91. 2*s.* 6*d.* How much did I originally invest?

(19) I invested in the 3 per cents. at $89\frac{1}{2}$, brokerage $\frac{1}{8}$. How much stock did I purchase, and what was the broker's commission, if I paid for investment and commission £410?

(20) The issue price of certain railway shares was £50, to be paid in five instalments of £10 each, the first payment on application. After a "call" or second payment of £10, the shares stood at £1 per share premium. I then invested £756, and after paying a further call of £10, a dividend was declared of $8\frac{1}{2}$ % per annum on the paid-up capital. What is the amount of my dividend, and what interest do I get for my money?

(21) A friend lent me £475 at 4 %; but to raise the money he sold out 3 per cent. Consols when they were at $87\frac{5}{8}$ (brokerage $\frac{1}{8}$). I kept the money three months, and meanwhile the funds rose to $91\frac{1}{2}$. How much have I to pay my friend to cover the interest and to replace the stock he previously held (brokerage again $\frac{1}{8}$)?

CHAPTER XI.

SQUARE AND CUBE ROOT.

§ 1. If a number be multiplied by itself, the product is called the SQUARE of that number, and the number is called the SQUARE ROOT of the product.

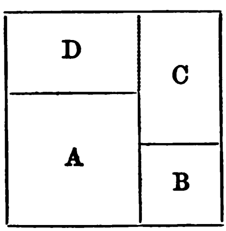
Roots:	1,	2,	3,	4,	5,	6,	7,	8,	9, &c.
Squares:	1,	4,	9,	16,	25,	36,	49,	64,	81, &c.

If this table were carried out ad infinitum, inspection would shew us the root of any proposed square. In absence of such a table, methods must be found to answer the same purpose.

§ 2. $1^2 = 1$; $10^2 = 100$; $100^2 = 10000$; $1000^2 = 1000000$, &c.; from which we see that a power of ten is squared by doubling its ciphers. Again, any number between 100 and 1000 will have a square between 10000 and 1000000, and will therefore have either five or six figures; or, generally:

Numbers with	1 figure	have in their squares	1 or 2	figures.
„	2 figures	„	3 or 4	„
„	3	„	5 or 6	„
„	4	„	7 or 8	„
	⋮		⋮ ⋮	
„	n	„	$2 \times n - 1$, or $2 \times n$	figures.

§ 3. The expression, "the square" of a *number*, is derived from the fact that to find the area of a square *surface* we multiply the number of units of length in one side by itself, the result being square units.



Suppose the whole side of a square of 8 units to be divided into

two parts of 5 and 3 units respectively, then the whole square will have 64 square units, while the sum of the two smaller squares, A and B, will be $25 + 9$, or 34 square units. This differs from the whole by 30 square units, which is consequently the area of the two surfaces C and D; and as each of these surfaces is 5 units long and 3 units broad, and therefore has 15 square units, we see that this is correct.

§ 4. Square the number 37.

$$\begin{aligned} 37 \times 37 &= (30 \times 37) + (7 \times 37) \\ &= (30 \times 30) + (30 \times 7) + (7 \times 30) + (7 \times 7) \\ &= (30 \times 30) + 2 \times (30 \times 7) + 7 \times 7; \\ \text{or } 37^2 &= 30^2 + 2 \times 30 \times 7 + 7^2; \\ 1369 &= 900 + 420 + 49. \end{aligned}$$

Similarly, $8^2 = 5^2 + 2 \times 5 \times 3 + 3^2$, as in the illustration of § 3; and, generally, if $A = a + b$, $A^2 = a^2 + 2 \times a \times b + b^2$.

Learn by heart: *The square of the sum of two numbers = the sum of their squares + twice their product.*

§ 5. We may also notice that if one of the two numbers be the greater, the product of the two numbers is greater than the square of the less, and still more is twice the product greater than the square of the less.

§ 6. Find the square root of 4096. From § 2, we find that as the square has 4 figures, the root has 2 figures, or, which is the same thing, the root is between 10 and 100.

The square of	10	is	100
„	20	„	400
„	30	„	900
„	40	„	1600
„	50	„	2500
„	60	„	3600
„	70	„	4900

whence it appears that the root lies between 60 and 70.

Subtract, then, 60^2 from 4096; remainder, 496.

$$\text{A} \qquad \begin{array}{r} 4096 \\ \underline{3600} \\ 496 \end{array}$$

This remainder, now, contains not only the square of the part of the root yet wanting, but also 2×60 (or 120) times that part; and as this last is much the larger of the two quantities (§ 5), we may, to begin with, disregard the square of the number sought, and try to find that number itself by dividing 496 by 120.

$$\begin{array}{r} 120)\,496\,(4 \\ 16 \end{array}$$

We obtain the quotient 4, as a guess at the second part of the root. Trial:

$$\begin{array}{lr} & 496 \\ \text{B} \qquad 120 \times 4 = & \underline{480} \\ & 16 \\ \text{C} \qquad 4 \times 4 = & \underline{16} \\ & 0 \end{array}$$

In A, then, we have subtracted 60^2; in B, $2 \times 60 \times 4$; and in C, 4^2; consequently, we have altogether subtracted $(60^2 + 2 \times 60 \times 4 + 4^2)$, or $(60 + 4)^2$, or 64^2, and as there is no remainder, $64^2 = 4096$, $\therefore$ 64 is the required square root. In other words, if the two parts of the root are respectively a and b, we must subtract from the square $a^2 + 2 \times a \times b + b^2$ to entitle us to say that we have subtracted $(a + b)^2$ (§ 4).

Find the square root of 339889. The root lies between 100 and 1000 (§ 2). Trial shews that it also lies between 500 and 600. Call 500 the first part of the root, or a.

$$\begin{array}{rr} & 339889 \\ a^2 = 500^2 = & \underline{250000} \\ & 89889 \\ 2 \times a \times b = 2 \times 500 \times 80 = & \underline{80000} \\ & 9889 \\ b^2 = 80 \times 80 = & \underline{6400} \\ & 3489 \end{array}$$

Now the second part, or b, must be contained more than 2×500,

($=1000$) times. Dividing, then, the remainder 89889 by 1000, we obtain for quotient 89; therefore the tens' figures is 8. Performing the two subtractions, we obtain for remainder 3489, and we have now subtracted 580^2. The third part of the root, or the units' figure, must be contained in the last remainder more than 2×580 ($=1160$) times. Dividing 3489 by 1160, or considering 580 as our new a, we find 3 for the units' figure.

$$\begin{array}{rr}
 & 339889 \\
a^2 = 580^2 = & \underline{336400} \\
 & 3489 \\
2 \times a \times b = 2 \times 580 \times 3 = & \underline{3480} \\
 & 9 \\
b^2 = 3^2 = & \underline{9} \\
 & 0
\end{array}
\qquad 1160)\,3489\,(3$$

Now we have subtracted 583^2 with remainder 0, and 583 is the required square root.

Find the square root of 12866569.

1st step. The root lies between 3000 and 4000. Subtract 3000^2.

$$\begin{array}{rl}
 & 12866569 \\
 & \underline{9000000} \\
6000) & 3866569\,(3000 \\
 & \underline{3250000}\quad 5 \\
7000) & 616569\,(80 \\
 & \underline{566400} \\
7160) & 50169\,(7 \\
 & \underline{50169} \\
 & 0
\end{array}$$

2nd step. Divide 3866569 by 2×3000, and it appears that the hundreds' figure is 6. Subtract $2 \times 3000 \times 600 + 600 \times 600 = 3960000$. As this, however, is more than the remainder, it cannot be subtracted, whence we see that 6 is too large a figure for the hundreds' place. Take 5, and subtract $2 \times 3000 \times 500 + 500 \times 500 = 3250000$.

3rd step. Divide 616569 by $2 \times (3000 + 500) = 2 \times 3500$. This gives 8 as the tens' figure. Subtract $2 \times 3500 \times 80 + 80 \times 80 = 566400$.

4th step. Divide 50169 by $2 \times (3000 + 500 + 80) = 2 \times 3580 = 7160$. This gives 7 as the units' figure. Subtract $2 \times 3580 \times 7 + 7 \times 7 = 50169$, and we have no remainder, and the square root required is 3587.

EXHIBITION OF THE SUCCESSIVE STEPS.

$$^{*}\sqrt{12866569} = \overset{a}{3000} + \overset{b}{500} \,\|\, \overset{a}{3500} + \overset{b}{80} \,\|\, \overset{a}{3580} + \overset{b}{7}$$

1st step ...	$a^2 =$ 3000^2	9000000	
2nd step ...	$2 \times a \times b =$ 3000000	3866569	
	$b^2 =$ 250000	3250000	
3rd step ...	$2 \times a \times b =$ 560000	616569	
	$b^2 =$ 6400	566400	
4th step ...	$2 \times a \times b =$ 50120	50169	
	$b^2 =$ 49	50169	*Ans.* 3587.
		0	

This process may be contracted by the following considerations and methods:

a. In the 2nd step we had to take 500 first 2×3000 times, and then 500 times; in all, we had to subtract $500 \times (2 \times 3000 + 500)$ which could have been done at once by subtracting it 6500 times. In the 3rd step, we had to subtract 80 first 2×3500 times, and then 80 times, or altogether 7080 times. In the 4th step, 7 had to be subtracted $2 \times 3580 + 7$ times, or altogether 7167 times.

b. The multiplication and subtraction can be done together as in division.

c. By substituting dots for ciphers, it will become evident that the figures will receive their true values by mere juxtaposition.

First contraction.

6...	$\sqrt{12866569}$ =	3...
5..	9......	5..
65.. × 5..	3866569	8.
70..	325....	7
8.	616569	3587
708. × 8.	5664..	
716.	50169	
7	50169	
7167 × 7	.....	

Second contraction.

$$\sqrt{12'86'65'69} = 3587$$

65	386
708	6165
7167	50169
	.....

* The symbol $\sqrt{}$ represents *r*, the initial letter of the word *radix*, root.

RULE :

a. Mark off the given number into "periods" of two digits, beginning with the units' figure (§ 2).

b. Find the nearest square root [3] below the 1st period [12], and the figure thus found is the first digit of the answer.

c. Subtract the square of this number [9] from the first period [leaving remainder 3], and bring down the next period [86].

d. Double the first figure of the root and use this [6] as the trial divisor.

e. Divide all the figures but the last in the second line [38] by this trial divisor; place the quotient [5] to the right both of the part of the root already found [35] and of the divisor [65]; multiply this by the new figure [5] and subtract [remainder 61].

f. Bring down the next period [65]; double the part of the root already found [35], and again use the result [70] as trial divisor.

g. Divide all the figures but the last in the third line [616] by this trial divisor, and write the quotient [8] to the right both of the root [358] and of the trial divisor [708]; multiply this by the new figure found [8], and subtract, &c.

Find the square root of 1144992021849.

	√1′14′49′92′02′18′49 = 1070043
20*7	14 49
214004	92 02 18
2140083	6 42 02 49
	• • • •

EXERCISE LXIII.

Find square roots of:

(1) 4.	(8) 289.	(15) 285970396644.
(2) 49.	(9) 3249.	(16) 501264.
(3) 100.	(10) 15129.	(17) 1607448649.
(4) 900.	(11) 582169.	(18) 41605800625.
(5) 160000.	(12) 61009.	(19) 9610000.
(6) 25000000.	(13) 956484.	(20) 123454321.
(7) 169.	(14) 68492176.	(21) 2892816758847744.

* When the trial divisor gives a 0 for quotient, care must be taken before bringing *down* the next period to place a cipher both in the root and in the trial divisor.

(22) A general arranges 8649 men in a solid square; how many men are there in each line?

(23) What is the length of the side of a square field 10 acres in extent?

§ 7. The square root of a fraction is found by finding the square root of the numerator and that of the denominator. For $\frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$, ∴ $\sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}$.

§ 8. Find the square root of 4317.

$$\begin{array}{lr} & \sqrt{43'17} = 65 \\ 125 & 7\,17 \\ & 92 \text{ over.} \end{array}$$

From this it appears that 65^2 is less while 66^2 is greater than 4317. The square root then lies between 65 and 66.

Exercise LXIV.

(1) A miser wished to arrange 5000 sovereigns in a square. What would be the length of each side, and how many sovereigns would be over?

(2) How many more sovereigns would he want to have one more sovereign in each side of the square?

(3) What must be subtracted from each of the following numbers to leave for remainder the greatest square each contains: 8000, 80000, 3492, 75912, 25601.

(4) Find the next exact square below 56135, 82060, 10000000, 123456789, 777777, 4853741.

§ 9. Find $\sqrt{7}$.

It lies between 2 and 3. Error, if either be taken, < 1.

Try $2\frac{1}{2}$; $\frac{5}{2} \times \frac{5}{2} = \frac{25}{4} = 6\frac{1}{4}$; too little.

Try $2\frac{3}{4}$; $\frac{11}{4} \times \frac{11}{4} = \frac{121}{16} = 7\frac{9}{16}$; too much.

∴ $\sqrt{7}$ lies between $2\frac{1}{2}$ and $2\frac{3}{4}$. Error, if either be taken, $< \frac{1}{4}$.

Try $2\frac{5}{8}$; $\frac{21}{8} \times \frac{21}{8} = \frac{441}{64} = 6\frac{57}{64}$; too little.

Try $2\frac{11}{16}$; $\frac{43}{16} \times \frac{43}{16} = \frac{1849}{256} = 7\frac{57}{256}$; too much.

∴ $\sqrt{7}$ lies between $2\frac{5}{8}$ and $2\frac{11}{16}$. Error, if either be taken, $< \frac{1}{16}$; and so on to within any assigned degree of accuracy.

§ 10. Instead of approximating by binary fractions, it is more convenient to work with decimals.

$\sqrt{7}$ lies between 2 and 3. Error, if either be taken, < 1.

Call $\sqrt{7} = \sqrt{\frac{700}{100}}$* $= \frac{\sqrt{700}}{10}$ (§ 7).

	$\sqrt{7'00'} = 26$
46	3 00
	24

$\therefore \frac{\sqrt{700}}{10}$ lies between 2·6 and 2·7. Error, if either be taken, < ·1.

Call $\sqrt{7} = \frac{\sqrt{70000}}{100}$.

	$\sqrt{7'00'00'} = 264$
46	3 00
524	24 00
	3 04

$\therefore \frac{\sqrt{70000}}{100}$ lies between 2·64 and 2·65. Error, " < ·01; and so on to any assigned degree of accuracy.

Mod. op.:

Find $\sqrt{7}$ to within $\frac{1}{1000000}$, i.e. to 6 places.

Call $\sqrt{7} = \frac{\sqrt{7000000000000}}{1000000}$.

	$\sqrt{7'00'00'00'00'00'00'} = 2645751$
46	3 00
524	2400
5285	30400
52907	397500
529145	2715100
5291501	6937500
	1645999

Ans. 2·645751.

The twelve ciphers might obviously be omitted from the first line and yet brought down two at a time, care being taken to put the decimal point in the root before bringing down the first pair of ciphers.

Find $\sqrt{11\frac{25}{64}}$.

$$\sqrt{11\tfrac{25}{64}} = \sqrt{\tfrac{729}{64}} = \frac{\sqrt{729}}{\sqrt{64}} = \tfrac{27}{8} = 3{\cdot}375.$$

	$\sqrt{7'29'} = 27$
47	3 29
	...

* *Even* powers of ten are chosen as denominators because they are exact squares (§ 2).

or $\sqrt{11\frac{25}{64}} = \sqrt{11 \cdot 390625}$.

$$\sqrt{11' \cdot 39'06'25} = 3 \cdot 375$$

63	2 39
667	5006
6745	33725
	.....

Ans. 3·375.

When the first pair of figures after the decimal point [39] was brought down, we were finding the root of $\frac{1139}{100}$, and hence our answer was $\frac{33}{10}$, i.e. 3·3; the next pair yielded hundredths, and so on. In marking off the periods, then, we must, as before, *begin from the units' place*, i.e. the decimal point, and mark both ways.

Find $\sqrt{57309\frac{15}{41}}$ to 5 places.

$\sqrt{57309\frac{15}{41}} = \sqrt{57309 \cdot 36585}$.

$$\sqrt{5'73'09' \cdot 36'58'53'} = 239 \cdot 39374...$$

43	1 73
469	4409
4783	18836
47869	448758
478783	1793753
4787867	35740465
47878744	222539685
	31024709
	&c.

This root cannot be conveniently found by vulgar fractions, the denominator 41 not being a square.

§ 11. The successive divisors continually become longer, in consequence of the addition of a figure to the right, which figure, however, becomes of less and less importance. Instead, then, of adding two figures to the remainder and one to the divisor, we may simply cut off one figure at a time from the divisor (p. 148), and proceed by simple division. We shall then get one place in the root for every figure in the divisor cut off. Accordingly, we may begin to cut off as soon as more than half the number of significant figures required have been obtained.

Find $\sqrt{\frac{1}{26}}$ to 12 places.

$\frac{1}{26} = \cdot 0\dot{3}8461\dot{5}$.

$\sqrt{\cdot 03'84'61'53'\ldots} = \cdot 196116135138$

29	284
386	2361
3921	4553
39221	63284
392226	2406361
3922321	5300553
[illegible]	1378232
	201535
	5419
	1497
	320
	6

Ans. ·196116135138.

Exercise LXV.

I. Find the square root to 4 places of:

(1) 3. (2) 19. (3) 11. (4) $5\frac{1}{8}$. (5) $7\frac{3}{10}$.

II. Find to 8 places the square root of:

(1) 2, ·2, ·02, ·002.

(2) 16, 1·6, ·16, ·016, ·0000016.

(3) $\cdot 1\dot{6}$, $\cdot\dot{1}\dot{6}$, $\cdot\dot{0}1\dot{6}$.

(4) $\cdot\dot{4}$, $\cdot\dot{1}9753086\dot{4}$, $\cdot 02\dot{7}$.

III. Find by vulgar fractions and by decimals to 4 places the square root of:

(1) $3\frac{1}{16}$. (2) $1\frac{64}{225}$. (3) $51\frac{21}{25}$. (4) $15\frac{10}{81}$.

IV. Find to 12 places the square root of:

(1) $\frac{5}{8}$. (3) $\frac{9}{1700}$. (5) $\frac{1}{3}$. (7) $\frac{1}{300}$. (9) $\frac{1}{4000}$.

(2) $\frac{7}{19}$. (4) $\frac{1}{2}$. (6) $\frac{1}{30}$. (8) $\frac{1}{400}$. (10) $\frac{5}{311}$.

V. Simplify to 3 places:

(1) $\sqrt{59} + \sqrt{\frac{7}{8}}$. (6) $\sqrt{2-\sqrt{2}}$

(2) $\sqrt{\cdot 2} - \sqrt{\cdot\dot{2}}$. (7) $\frac{1}{\sqrt{2}}$.*

(3) $\sqrt{144^2+17^2}$. (8) $\frac{2}{\sqrt{1\cdot 1}}$.

(4) $\sqrt{113^2-112^2}$. (9) $\frac{1}{\sqrt{\cdot 001}}$.

(5) $\sqrt{1+\sqrt{3}}$ (10) $\sqrt{5+\sqrt{5+\sqrt{5}}}$

* $\frac{1}{\sqrt{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \sqrt{\frac{1}{2}} = \sqrt{\cdot 5}$, which find.

VI. Find x from the following (3 places):

(1) $45 : x = x : 80$.

(2) $1 : x = x : 2$.

(3) $x : 20 = 245 : x$.

(4) $1\frac{13}{36}$, x, $8\frac{4}{7}$, three numbers in G. P. (p. 140).

Exercise LXVI.

The following require a slight knowledge of Geometry. **Results** to be found to 4 places:

(1) A rectangular room is 24 ft. long and 18 ft. broad. Find the diagonal of the floor.

(2) A tower is 180 ft. high; I stand 19 ft. away from the base. How far am I in a straight line from the top of the tower?

(3) The disc of a pendulum 85 inches long touches in its sweep two points, A and B, which are in the same horizontal line, and are 1 inch above the lowest position of the disc. How far are they apart?

(4) In the 3rd question, suppose the length of the pendulum to be 41 inches, and A and B to be 18 inches apart, how far are they above the lowest position of the disc?

(5) How long must a ladder be to reach to the top of a house 60 ft. high, when its foot is placed 11 ft. from the wall?

(6) Given in a right-angled triangle:

a. Hypotenuse 200, base 70; find perpendicular.

b. Hypotenuse 1, perpendicular $\frac{1}{\sqrt{2}}$; find base.

c. Base $\sqrt{3}$, perpendicular 1; find hypotenuse.

(7) The foot of a column 200 ft. high is 300 ft. from the base of a house 75 ft. high. Find the distance of the top of the column from the top of the house.

(8) Find the diagonal of a square whose side is 100 feet.

(9) Find the side of a square whose diagonal is 100 feet.

(10) Find the length of the perpendicular from the vertex to the base of an equilateral triangle whose side is 100.

(11) Find the side of an equilateral triangle, if the perpendicular from the vertex is 100.

(12) Find the side of a square equal to a rectangle whose sides are 588 and 507 feet.

§ 12. CUBE ROOT.

If the square of a given number be multiplied by the given number, the product is called the CUBE of the given number, and the given number is called the CUBE ROOT of the product.

$5^2 = 25$; $25 \times 5 = 125$; 5^3 or 125 is the cube of 5; and 5 is the cube root of 125, which is written $\sqrt[3]{125}$.

Roots	1,	2,	3,	4,	5,	6,	7,	8,	9.
Cubes	1,	8,	27,	64,	125,	216,	343,	512,	729.

§ 13.

$$1^3 = 1$$
$$10^3 = 1000$$
$$100^3 = 1000000$$
$$1000^3 = 1000000000,$$
&c., &c.

from which we see that a power of 10 is cubed by trebling its ciphers; and reasoning analogous to that in § 2 shews that:

Numbers with	1 figure	have in their cubes	1, 2 or 3	figures,
„	2 figures	„	4, 5 or 6	„
„	3 „	„	7, 8 or 9	„
	⋮		⋮	

Hence we shall mark off the given cube into periods of *three* figures, beginning with the units' figure.

§ 14. The expression, the cube of a number, is derived from the fact, that to find the volume of a *cube* we multiply the number of units in one side by itself, and the product again by the same number. (Part II. Ch. IV. § 6.)

§ 15. We have seen that if $A = a + b$, $A^2 = a^2 + 2 \times a \times b + b^2$ (§§ 3, 4). Similarly, it is shewn in books on Algebra, that $A^3 = a^3 + 3 \times a^2 \times b + 3 \times a \times b^2 + b^3$, or $a^3 + (3 \times a^2 + 3 \times a \times b + b^2) \times b$, thus:

$$\begin{aligned}
37^3 &= 37 \times 37 \times 37 = 37^2 \times 37 \\
&= (30^2 + 2 \times 30 \times 7 + 7^2) \times (30 + 7) \\
&= (30^2 + 2 \times 30 \times 7 + 7^2) \times 30 + (30^2 + 2 \times 30 \times 7 + 7^2) \times 7 \\
&= 30^3 + 2 \times 30^2 \times 7 + 7^2 \times 30 + 30^2 \times 7 + 2 \times 30 \times 7^2 + 7^3 \\
&= 30^3 + 3 \times 30^2 \times 7 + 3 \times 30 \times 7^2 + 7^3 \\
50653 &= 27000 + 18900 + 4410 + 343.
\end{aligned}$$

Hence to subtract 37^3 we shall have to subtract $30^3 + 3 \times 30^2 \times 7 + 3 \times 30 \times 7^2 + 7^3 = 30^3 + (3 \times 30^2 + 3 \times 30 \times 7 + 7^2) \times 7$, and after subtracting 30^3, we have from the remainder to subtract $7 \times (3 \times 30^2 + 3 \times 30 \times 7 + 7^2)$. Of the quantity in brackets, 3×30^2 is the largest term, and may therefore be used as trial divisor.

Find $\sqrt[3]{50653}$.

	$\sqrt[3]{50653}\,(30+7$
$3 \times 30^2 \quad = 2700$	27000
$3 \times 30 \times 7 = 630$	23653
$7^2 = 49$	23653
3379×7	.....

Here 2700, the trial divisor, is contained 7 times in the remainder 23653.

	Contraction.
$3 \times 30^2 \quad = 27..$	$\sqrt[3]{50'653} = 37$
$3 \times 30 \times 7 = 63.$	23 653
$7^2 = 49$	.....
3379	

Here the trial divisor 27 is contained at least* 7 times in all but the last two figures of the remainder (236).

Exercise LXVII.

Find the cube root of:

(1) 68921.	(4) 592704.	(7) 3375.
(2) 110592.	(5) 389017.	(8) 10648.
(3) 205379.	(6) 300763.	(9) 54872.

* A first inspection would lead us to try 8, but the addition of $3 \times 30 \times 8$ and 8^2 would have made the divisor too great.

Find the cube root of 42028039032·832.

	$\sqrt[3]{42'028'039'032\cdot 832} = 3000 + 400 + 70 + 6 + \cdot 8$	
$3000^3 = \ldots\ldots\ldots\ldots\ldots\ldots$	27	
	15 028	First remainder.
First trial divisor :		
$3 \times 3000^2 = 27\ldots\ldots$		
$3 \times 3000 \times 400 = 36\ldots\ldots$		
$400^2 = 16\ldots\ldots$		
$3076\ldots\ldots \times 4\ldots$	12 304	
	2 724 039	Second remainder.
Second trial divisor :		
$3 \times 3400^2 = 3468\ldots\ldots$		
$3 \times 3400 \times 70 = 714\ldots$		
$70^2 = 49\ldots$		
$353989\ldots \times 7.$	2 477 923 . . .	
	246 116 032	Third remainder.
Third trial divisor :		
$3 \times 3470^2 = 361227\ldots$		
$3 \times 3470 \times 6 = 6246.$		
$6^2 = 36$		
36185196×6	217 111 176	
	29 004 856·832	Fourth remainder.
Fourth trial divisor :		
$3 \times 3476^2 = 36247728$		
$3 \times 3476 \times \cdot 8 = 8342\cdot 4$		
$\cdot 8^2 = \cdot 64$		
$36256071\cdot 04 \times \cdot 8$	29 004 856·832	
	. . . .	*Ans.* 3476·8.

Remarks :

(1) The second figure of the root, 400, is found by dividing the 1st remainder, 15028039032·832, by the 1st trial divisor, 27000000, or, which is the same, by dividing 150. . by 27 ; but then the new figure is called 4 and not 400 ; similarly, the 3rd figure, 7, is found by dividing 27240. . by 3468 ; the 4th figure, 6, by dividing 290048. . by 36247728, and so on.

(2) The 1st trial divisor, 27, is 3 times the square of the 1st figure ; the 2nd, 3468, is 3 times the square of the part already *found*, 34. Now :

$$
\begin{aligned}
34^2 &= 30^2 + (2\times 4\times 30) + 4^2, \text{ and}\\
3\times 34^2 &= (3\times 30^2)+(3\times 2\times 4\times 30)+(3\times 4^2)\\
&=(3\times 30^2)+(2\times 3\times 4\times 30)+(3\times 4^2)\\
&=(3\times 30^2+3\times 4\times 30+4^2)+(3\times 4\times 30)+4^2+4^2\\
&= 3076 + 360 + 16 + 16.
\end{aligned}
$$

Hence this 3468 can be formed by adding the three preceding lines together and adding in another 4^2. Similarly, the third trial divisor 361227 is 3×347^2. Now:

$$
\begin{aligned}
347^2 &= 340^2 + (2\times 340\times 7) + 7^2, \text{ and}\\
3\times 347^2 &= (3\times 340^2)+(3\times 2\times 340\times 7)+(3\times 7^2)\\
&=(3\times 340^2)+(2\times 3\times 340\times 7)+(3\times 7^2)\\
&=(3\times 340^2+3\times 340\times 7+7^2)+(3\times 340\times 7)+7^2+7^2\\
&= 353989 + 7140 + 49 + 49,
\end{aligned}
$$

and can therefore be obtained by adding another 49 to the three lines last obtained.

(3) Having found the new figure of the root by the help of a trial divisor, the line below that trial divisor is found by multiplying the old part of the root by 3 times the new figure. This can be done in one line unless the new figure be 8 or 9, in which case the old part must be first multiplied by 3 aside. (Part I. p. 98.)

Contracted form.

$$
\begin{array}{r}
27\\
36\\
16\\
\hline
3076\\
16\\
\hline
3468\\
\hline
714\\
49\\
\hline
353989\\
49\\
\hline
361227\\
\hline
6246\\
36\\
\hline
36185196\\
36\\
\hline
36247728\\
83424\\
64\\
\hline
3625607104
\end{array}
\qquad
\begin{array}{l}
\sqrt[3]{42'028'039'032\cdot832}=3476\cdot8\\
\;15028\\
\;\;2724039\\
\;\;\;246116032\\
\;\;\;\;29004856832\\
\;\;\;\;\cdots\cdots\cdots\cdots
\end{array}
$$

Find the cube root of 127268840262·941343.

7500	$\sqrt[3]{127'268'840'262\cdot941'343} = 5030\cdot07$
450	2 268 840
9	5 313 262 941 343
754509	••••••• •••••
9	
7590270000	
1056300	
49	
759037563049	

1st trial divisor, 75; 1st remainder, 2268; 75 in 22.., 0, which we place after 5 in the root. The first part of the root is now 50, and $3 \times 50^2 = 7500$. Hence a 0 in the root requires 00 in the trial divisor.

§ 16. To find the cube root of a fraction, reduce the fraction to a decimal to 3 times the required number of places, unless the denominator happen to be obviously an exact cube, in which case find cube roots of numerator and denominator separately.

Find cube root of $\frac{1}{50}$ to 4 places.

12	$\sqrt[3]{\cdot020} = \cdot2714\ldots\ldots$
42	12000
49	317000
1669	97489000
49	9190056......
2187	
81	*Ans.* ·2714.
1	
219511	
1	
220323	
3242	
16	
22064736	

EXERCISE LXVIII.

I. Find the cube root of:

(1) 884736.

(2) 40353607.

(3) 1191016.

(4) 8108486729.

(5) 115145914625.

(6) ·017173512.

(7) ·000000004096.

(8) ·000064481201.

II. Find the cube root (to 4 places) of:

(1) $\frac{343}{512}$. (3) $\frac{12}{27}$. (5) $\frac{5}{6}$. (7) $\frac{6}{7}$. (9) ·01.
(2) $\frac{1}{729}$. (4) $\frac{41}{32}$. (6) $15\frac{2}{3}$. (8) ·1. (10) ·001.

III. (1) If 12167 dice are piled up into a solid cube, how many dice will there be in each edge?

(2) Find the area of the surface of a cube whose volume is 3 cubic yds., 10 ft., 216 in.

§ 17. Properties of Squares and Cubes.

Some integers have square roots which are themselves integers, such as 4, 9, 16...1024, &c. (§ 1). The question arises whether the numbers between any two of these squares have a fractional square root, or no square root at all. Take any number between 4 and 9, say 6; its square root, if it have any, lies between 2 and 3, and must consequently be an improper fraction, say $\frac{a}{b}$. Now $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$; since a and b are prime to one another, we have in squaring the terms introduced no common factor, and $\therefore \frac{a^2}{b^2}$ is also at lowest terms and consequently fractional, and $\therefore$ not 6. Hence the square of a fraction cannot be integral, or, which is the same thing, an integer cannot have a fractional square root.

The number of exact squares is infinite; but there are within any assigned limits many more numbers *not* having exact square roots than there are of exact squares.

These remarks (*mutatis mutandis*) apply also to exact cubes.

§ 18. The square of a number ending in 1 or 9 ends in 1, $\because 1 \times 1 = 1$ and $9 \times 9 = 81$.

The square of a number ending in		2 or 8	ends in	4.
„	„	3 or 7	„	9.
„	„	4 or 6	„	6.
„	„	5	„	5.
„	„	0	„	00.

Hence no exact square can end in 2, 3, 7, 8, or an odd number of ciphers.

A cube may end in any digit, but if it end in 0, it must end in a number of ciphers divisible by 3.

The cube of a number	ending in	1	ends in	1,
"	"	2	"	8,
"	"	3	"	7,
"	"	4	"	4,
"	"	5	"	5,
"	"	6	"	6,
"	"	7	"	3,
"	"	8	"	2,
"	"	9	"	9.

§ 19. Terminating decimals can only be squares if the number of places be even, and then they follow the rule of § 18.

Terminating decimals can only be cubes if the number of places be divisible by 3.

§ 20. Recurring decimals may or may not be exact squares or cubes; thus:

$$\sqrt{\dot{4}} = \sqrt{\tfrac{4}{9}} = \tfrac{2}{3}, \text{ exact.}$$

$$\sqrt{\dot{5}} = \sqrt{\tfrac{5}{9}} = \frac{\sqrt{5}}{3}, \text{ not exact.}$$

$$\sqrt[3]{\dot{2}9\dot{6}} = \sqrt[3]{\tfrac{296}{999}} = \sqrt[3]{\tfrac{8}{27}} = \tfrac{2}{3}, \text{ exact.}$$

$$\sqrt[3]{\dot{2}9\dot{7}} = \sqrt[3]{\tfrac{297}{999}} = \sqrt[3]{\tfrac{33}{111}}, \text{ not exact.}$$

§ 21.

If $a > 1$, $a^2 > a$, and $a^3 > a^2$; $\therefore$ if $b > 1$, $\sqrt{b} < b$, $\sqrt[3]{b} < \sqrt{b}$.

If $a < 1$, $a^2 < a$, and $a^3 < a^2$; $\therefore$ if $b < 1$, $\sqrt{b} > b$, $\sqrt[3]{b} > \sqrt{b}$.

§ 22. If an exact square end in 6, the tens' figure is odd; if in any other figure, the tens' figure is even.

§ 23. If any number is not an exact square we can find two numbers differing from one another by as small a quantity as we please, of whose squares one shall be greater and the other less than the given number. This we have called finding the square root to *so many* places of decimals. For example:

$\sqrt{2}$ lies between	1·0	and 2·0	to the nearest integer,
"	1·4	and 1·5	to one place,
"	1·41	and 1·42	to two places,
"	1·414	and 1·415	to three "
"	1·4142	and 1·4143	to four "

and so on.

§ 24. Here and in § 9 we have in our expressions tacitly assumed that there are such things as $\sqrt{2}$, $\sqrt{7}$, &c., although they are not expressible by exact numbers. Such language is justified by the following considerations, which will be understood by those who have been able to work Exercise LXIV.

If the side of a square be 1, the square on the diagonal is 2 (Euc. I. 47), and the length of the diagonal is $\sqrt{2}$. $\sqrt{2}$, then, is an actual quantity, and measurements of the line continually increasing in accuracy by stages of the decimal scale will yield the above approximations.

Similarly, lines can be found whose length is $\sqrt{3}$, $\sqrt{7}$, &c.

§ 25. $3^4 = 3 \times 3 \times 3 \times 3 = 3^2 \times 3^2 = (3^2)^2$. Hence the square root of the square root is the 4th root, or $\sqrt[4]{a} = \sqrt{\sqrt{a}}$.

$$3^6 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^2 \times 3^2 \times 3^2 = (3^2)^3.$$

Again, $3^6 = 3^3 \times 3^3 = (3^3)^2$, $\therefore \sqrt[6]{a} = \sqrt[3]{\sqrt{a}} = \sqrt{\sqrt[3]{a}}$. Hence the cube root of the square root, or the square root of the cube root, is the 6th root. Similarly, the cube root of the cube root is the 9th root; the cube root of the 4th root, or the 4th root of the cube root, is the 12th root; and so on.

§ 26. Find $\sqrt{63361}$ in the septenary scale.

$$\begin{array}{ll} & \sqrt{6'33'61} = 236 \\ 43 & \quad 2\,33 \\ 466 & \quad 41\,61 \\ & \quad \cdots\cdot \end{array}$$

Ans. 236 septenary.

Find $\sqrt{234}$ (quinary) to three places of quinals.

$$\begin{array}{ll} & \sqrt{2'34} = 13{\cdot}123 \\ 23 & \quad 134 \\ 311 & \quad 1000 \\ 3122 & \quad 13400 \\ 3124 & \quad 2101 \end{array}$$

Ans. 13·123.

CHAPTER XII.

VARIOUS.

§ 1. Interconversion of Fractions in different Scales.

Reduce $\cdot\dot{1}2\dot{3}$ (quinals) to duodecimals (to 6 places).

$$s = \quad \cdot 123123\ldots\ldots$$
$$5^3 \times s = 123\cdot 123123\ldots\ldots$$

$(5^3 - 1) \times s = 444 \times s = 123 \qquad s = \frac{123}{444}$ in the quinary scale,

$= \frac{38}{124} \Big| \frac{19}{62}$ in the decimal scale.

$$\begin{array}{lll}
\text{Dec.} & \text{Dec.} & \text{Duodec.} \\
62) & 19 \times 12 & (\cdot 38166e\chi \\
& \overline{228} & \\
& 42 \times 12 & \\
& \overline{504} & \\
& 8 \times 12 & \\
& \overline{96} & \\
& 34 \times 12 & \\
& \overline{408} & \\
& 36 \times 12 & \\
& \overline{432} & \\
& 60 \times 12 & \\
& \overline{720} & \\
& 38 \times 12 & \\
& \overline{456}, \&c. &
\end{array}$$

Ans. $\cdot 38167\chi$.

§ 2. Duodecimals.

The Duodecimal scale, both integral and fractional, can be usefully applied to the calculation of small areas and volumes. The

foot is taken as the unit; the inch, $\frac{1}{12}$ of 1 foot, is in the *first* place, and the "part" or 12th of an inch, or $\frac{1}{12^2}$ of 1 foot, in the *second.* Hence in square measure, the square foot being the unit, the square inch is in the second place, and the square part in the fourth; and in cubic measure, the cubic foot being the unit, the cubic inch is in the third place, and the cubic part in the sixth place.

Given length, 1 ft., 11 in., 3 pts.; breadth, 1 ft., 7 in., 6 pts.; find area. $1{\cdot}e3 \times 1{\cdot}76$. (Part I. Ch. X.)

$$\begin{array}{r}
1\,e\,3\\
\underline{1\,7\,6}\\
e\,7\,6\\
1\,1\,6\,9\\
\underline{1\,e\,3}\\
3{\cdot}1\,9\,4\,6
\end{array}
\begin{array}{l}
\\ \\ \\ \\ \\
=3 \text{ sq. ft., } (1\times 12+9) \text{ sq. in., } (4\times 12+6) \text{ sq. pts.}\\
=3 \text{ sq. ft., } 21 \text{ in., } 54 \text{ pts.; } \tfrac{54}{144}\big|\tfrac{3}{8},\\
=3 \text{ sq. ft., } 21\tfrac{3}{8} \text{ sq. in.}
\end{array}$$

or by contracted multiplication, keeping 2 places, to be correct to square inches.

$$\begin{array}{r}
1{\cdot}e\,3\\
\underline{6\,7\,1}\\
1\,e\,3\\
1\,1\,7\\
\underline{1\,0}\\
3{\cdot}1\,t
\end{array}
=3 \text{ sq. ft. } 22 \text{ in.} \quad \text{Error, } \tfrac{5}{8} \text{ of } 1 \text{ sq. in.}$$

§ 3. International Calculations.*

a. Length.

	Multiply by:	or Divide by:
To turn yards into metres	·9143862	1·09363
,, ,, Prussian ells	1·371	·7294
,, ,, Austrian ells	1·1743	·8516
,, ,, Spanish varas	1·0784	·9273
,, ,, Portuguese varas	·8318	1·2022
,, ,, Russian arsheens	1·2857	·7
,, miles into kilometres	1·609315	·6213824
,, ,, Prussian miles.........	·21364	4·6807
,, ,, Austrian miles.........	·21212	4·7142

* These calculations are mainly based on Woolhouse's Measures, Weights and Moneys of all Nations.

	Multiply by:	or Divide by:
To turn miles into Spanish leagues	·23723	4·2152
" " Portuguese miles......	·7821	1·2786
" " Russian versts	1·50852	·6629
b. Surface.		
To turn sq. yds. into centiares (sq. metres)	·83612	1·196
" acres into hectares	·404671	2·471143
" " Prussian morgen	1·5848	·631
" " Austrian joch	·70308	1·4223
" " Portuguese geiras......	·69187	1·4453
c. Capacity.		
To turn gallons into litres..................	4·54345	·220097
" " Prussian eimer	·06614	15·118
" " Austrian eimer......	·08027	12·4572
" " Spanish cantaros ...	·28264	3·538
" bushels into hectolitres	·363476	2·751211
" " Austrian metzen ...	·15762	6·3442
" " American bushels...	1·03152	·96944
" quarters into Russian chetverts .	1·3863	·7213
d. Weight.		
To turn lbs. av. into lbs. troy	1·21527	·82285714
" grains into grammes	·06479895	15·4323487
" lbs. av. into kilogrammes.........	·45359265	2·2046212
" " Prussian pounds ...	·96983	1·0311
" " Austrian pounds ...	·80959	1·2352
" " Spanish pounds ...	·9858	1·0144
" " Portuguese pounds .	·98828	1·01186
" " Russian pounds ...	1·10786	·90264
" cwts. into quintals (metric) ...	·508023765	1·9684118
" " Prussian zentner ...	·987428	1·012732
" " Austrian zentner ...	·90674	1·102857
" " Portuguese quintals	·864745	1·1564107
" " Spanish quintals ...	1·10414	·90571
" " Russian berkowitz .	3·102	·32237
" tons into French tons (milliers)	1·01604753	·9842059
" " Russian packen	2·068008	·483557

§ 4. USE OF THE TABLES.

Express 9 tons, 13 cwt., 1 qr., 16 lbs., as metric tons or milliers.

9 tons, 13 cwt., 1 qr. (£9. 13*s*. 3*d*.) = 9·6625 tons.
16 lbs. ($\frac{1}{7}$ of ·05) = ·007142...
9·669642 ,,

Multiply by 1·015652:

```
1·015652
46 9669
--------
 91409
  6094
   609
    91
     6
--------
9·8209
     1
```

Ans. 9·821 milliers. Error < 1 kilog.

or divide by ·9846875:

```
·9846875) 9·669642 (9·820
          807454
           19704
              10
```

Ans. 9·820.

For conversion of foreign weights and measures to English, *multiply* by the given divisors or *divide* by the given multipliers, i.e. reverse the processes.

Express 419·785 kilometres as miles.

Multiply by ·6213824:

```
419·785
4283 126
--------
2518710
  83957
   4198
   1259
    335
      8
      2
--------
260·8469 × 8
  6·775  × 20
  15·5 ·· × 11
 170·5
```

Ans. 260 m., 15 fur., 170½ yds.

or divide by 1·609315:

```
1·609315) 419·7850 (260·8470
           97 9220        8
            1 3631   ------
               757   6·776
               113      20
                 1   ------
                     15·52·
                        11
                     ------
                     170·7
```

Ans. 260 m., 15 fur., 170$\frac{7}{10}$ yds.

Error < 7 inches.

§ 5. COINAGE.

The ratios for interconversion of the coinage of different nations can only be given at par, and would accordingly be nearly useless, as that rate rarely prevails. If the exchanges* are given, Chain Rule will apply.

* By "exchange" is meant the rate at which bills due in foreign countries are negotiable here, which fluctuates with the state of trade, &c.

Find the value of £647. 11*s.* 1*d.* in reis (Portugal), at 57½*d.* per milrei.

x	647·5541$\dot{6}$
·23958$\dot{3}$	1

2·39583333) 6475·5417 (2702·835
1683 8750
6 7917
2 0000
834
115

Ans. 2702·835 reis = 2702 milreis, 835 reis.

Find the arbitrated exchange* between London and Lisbon, if 1 milrei = 5·95 fr., and £1 = 25·15 fr.

£x	5·95 fr.
fr. 25·15	£1

25·15) 5·950 (·236
920
165
14

Ans. 4*s.* 8¾*d.*, or 56¾*d.* per milrei.

§ 6. Find the price per yard in English money at 7·85 fr. per metre; also at 4·27 fr., at 5·19 fr., and at 6·45 fr. per metre; exchange, 25·75.

To turn metres into yards, we multiply by 1·09363. Multiplying, then, this ratio by the exchange, we obtain the common divisor for the several prices given.

25·75
363 901
25750
2318
77
15
1
28·161

28·161) 7·850 (·2787
2 218 9
247
22 *Ans.* 5*s.* 7*d.*

28·161) 4·270 (·1516
1 454
46
18 *Ans.* 3*s.* 0¼*d.*

28·161) 5·190 (·1843
2 374
121
9 *Ans.* 3*s.* 8¼*d.*

28·161) 6·450 (·229
818
225
2 *Ans.* 4*s.* 7*d.*

* The "arbitrated exchange" is the rate realized by remitting to a place abroad, *not directly*, but *viâ* some other place or places.

Thus common multipliers or divisors can, as occasion requires, be established for the working of classes of questions.

§ 7. For ordinary purposes francs are valued at 25 per £, which leads to easy calculations.

The following are ratios compounded on the Metric System, the money being calculated *at par* (fr. 25·2215 = £1), i.e., according to the intrinsic value of the coins.

To find in one operation the price in English money:

					Multiply by: or	Divide by:
a.	Per yard,	given price per	metre in	francs	·03625	27·5862
b.	„ sq. yard,	„	sq. metre	„	·03315	30·1649
c.	„ acre,	„	hectare	„	·016044	62·3275
d.	„ gallon,	„	litre	„	·1765735	5·66335
e.	„ bushel,	„	hectolitre	„	·0144113	69·3897
f.	„ lb. av.,	„	kilo	„	·01798432	55·6038536
g.	„ cwt.,	„	quintal	„	·020142488	49·646298
h.	„ ton,	„	millier	„	·040282976	24·823149

Find the cost in English money of 1 yd. at fr. 7·72 per metre.

$$\begin{array}{r} \cdot 03625 \\ \underline{277} \\ 2538 \\ 254 \\ \underline{7} \\ \cdot 2799 \end{array}$$

Ans. 5*s.* 7¼*d.*

Exercise LXIX.

(1) Express 73 lbs., 12 oz. av. in:

a. Kilogs. *c.* Austrian pounds.
b. Russian pounds. *d.* Prussian pounds.

(2) Express 25·625 French tons as:

a. English tons. *b.* Russian packen.

(3) Find the cost in English money per yd. at fr. 2·45 per metre.

(4) „ „ per cwt. at fr. 16·75 per quintal.

(5) „ francs per French ton at 14*s.* 9*d.* per Engl. ton.

(6) How many litres in 4½ Austrian eimer?

(7)* The average height of the barometer at Paris is 76 centimetres. Reduce this to inches correctly to three places of decimals.

* Nos. 7 to 20 are copied from an excellent tract on the Metric System, by J. J. Walker, Esq., M.A.

(8) Reduce the hectare to a. r. sq. po. yds. exactly. How much does it differ from $2\frac{1}{2}$ acres?

(9) Supposing the quadrant of the meridian of Paris to be 6213 m., 6 fur., 23 po., 4 yds. in length, calculate the length of the metre in inches to five places.

(10) The French post-office allows 7·5 grammes for a single postage; the English $\frac{1}{4}$ of an oz. av. By how many grains does the French exceed the English allowance?

(11) How many hectolitres = 1 cubic metre? A tank is $37\frac{1}{2}$ decimetres long, by 25 wide and 18 deep. How many gallons would it hold?

(12) If wine be sold at 457 francs the cask of 7 hectolitres, what would be the corresponding price in *s. d.* of the bottle of 6 to the gallon?

(13) How many bushels = 1 hectolitre? If wheat be sold at 29 francs the hectolitre, what would be the corresponding price per bushel in *s. d.*?

(14) The length of the tunnel through Mont Cenis will be about 12·22 kilometres. What will this be in miles?

(15) The diameter of bore and weight of a piece of French ordnance are given as 27 centimetres and 22,000 kilogrammes. Give the corresponding measure and weight in inches and cwt.

(16) A building plot in Paris is offered for sale at 75 francs per sq. metre. What would be the corresponding price per sq. yard in £. *s. d.*?

(17) The distance between two stations on a Belgian railway is set down as $7\frac{1}{2}$ kilometres, and is done by a train in 12 minutes. What is the rate per hour in miles?

(18) The pressure of the atmosphere at the average height of the barometer is $14\frac{3}{4}$ lbs. av. to the sq. inch. What would be the corresponding pressure in kilogrammes to the square centimetre?

(19) If the sack of flour of 157 kilogrammes be sold at 53·75 fr., what would be the corresponding price of the quartern (4 lbs.) in *s.d.*?

(20) The rent of a farm of 23·25 hectares is 1225 francs. What is the rate per acre in £. *s. d.*?

CHAPTER XIII.

ARITHMETICAL COMPLEMENTS.*

§ 1. The ARITHMETICAL COMPLEMENT of a number is the quantity by which it falls short of the next power of 10 above it, or, in other words, the DEFECT of the number from that power of 10. Thus the A. C. or defect of 3792 is $10000 - 3792 = 6208$. (Cf. Part I. Ch. I. § 11.)

§ 2. This defect is obtained by complementing to 9 each digit but the units' figure, which must be complemented to 10. By its means several arithmetical processes can be performed in shapes very different from those in common use, and these are useful when the A. C. is small. The same principle can be extended to cases where complementing only the right-hand portion of a number will yield a number not greater than 12 followed by ciphers.

§ 3. Addition can be performed by the aid of subtraction. Thus "to add 9, add ten and *deduct* one." (Part I. p. 11, line 10.) Similarly, to add 3995, it is shortest to add 4000 and *subtract 5*. Conversely, subtraction can be performed by the aid of addition; thus to subtract 9, subtract 10 and *add* 1; and to subtract 3995, subtract 4000 and *add 5*.

§ 4. Multiplication, which is a series of additions (Part I. p. 44), can accordingly be performed by a series of subtractions, or by a multiplication and subtraction. Thus to multiply by 998, multiply by 2 and subtract the product from the multiplicand three places to the right. (Part I. p. 108.) Similarly, to multiply by 3995, we may multiply by 4 and subtract 5 times the multiplicand three places lower down.

$7243{\cdot}849516 \times 39{\cdot}95.$

```
        7243849516 × 4..
       289753 98064...
       289391·78816420        Ans. 289391·7881642.
```

* The matter of this chapter is mostly drawn from a pamphlet by George Suffield, Esq., M.A., Clare College, Cambridge.

$2345{\cdot}\dot{7}\dot{6} \times 79{\cdot}993.$

$$\begin{array}{r}
\underline{23457\dot{6}\dot{6}} \times 8\ldots. \\
08 \\
18766\dot{1}\dot{4}|1414 \\
1876449\,9382 \\
76
\end{array}$$

Ans. $187644{\cdot}9937\dot{6}$

Wording: 48′, carry 4; 56, 60′, carry 6; hence (Ch. VI. § 16) $08+6=\dot{1}\dot{4}'$, carry 6; 40, 46′, carry 4; 32, 36′, carry 3, &c. Put on 1414 for the (....) and subtract 7 times the multiplicand, thus: 42 and 2′, 44, carry 4; 49, 53, and 8′, 61, carry 6; hence (Ch. VI. § 16) $82-6=\dot{7}\dot{6}'$, carry 6; 35, 41, and 3′, 44, carry 4; 28, 32, and 9′, 41, carry 4, &c.

§ 5. Division, which is a series of subtractions (Part I. p. 76), can be performed by a series of additions.

Divide 123456789 by 7384. 7384 = 8000 − 616; hence for every 8000 deducted we have deducted 616 too much, which must therefore step by step be added to get the true remainder.

$$\begin{array}{lll}
 & 8000)\,123456789\,(166\left.\begin{matrix}\\ 1\end{matrix}\right\}\ 18\left.\begin{matrix}\\ 1\end{matrix}\right\} & \\
 & \quad 4\ldots & \\
1 \times 616\ldots\ldots & \quad 616 & \\
 & \quad 49616 & \text{1st remainder.} \\
 & \quad 1\ldots & \\
6 \times 616\ldots\ldots & \quad 3696 & \\
 & \quad 53127 & \text{2nd remainder.} \\
 & \quad 5\ldots & \\
6 \times 616\ldots\ldots & \quad 3696 & \\
 & \quad 8823 & \text{3rd remainder } (a) \\
 & \quad 0\ldots & \\
1 \times 616\ldots\ldots & \quad 616 & \\
 & \quad 14398 & \text{3rd remainder } (b) \\
 & \quad 6\ldots & \\
1 \times 616\ldots\ldots & \quad 616 & \\
 & \quad 70149 & \text{4th remainder.} \\
 & \quad 6\ldots & \\
8 \times 616\ldots\ldots & \quad 4928 & \\
 & \quad 11077 & \text{5th remainder } (a) \\
 & \quad 3\ldots & \\
1 \times 616\ldots\ldots & \quad 616 & \\
 & \quad 3693 & \text{5th remainder } (b)
\end{array}$$

Ans. 16719 times and 3693 over.

REMARKS:

(1) The successive remainders are the same as by the common method.

```
7384)123456789(16179
       49616
        53127
         14398
          70149
           3693
```

(2) 8000 in 12345, once and 4345 over, ∴ 7384 in 12345, once and $4345 + 1 \times 616 = 4961$ over; 8000 in 49616, 6 times and 1616 over, ∴ 7384 in 49616, 6 times and $1616 + 6 \times 616 = 5312$ over; and so on.

(3) This method gives for third remainder 8823, which contains 8000, and *a fortiori* 7384, one more time, altering the 6 in the quotient to 7, and the remainder, after an addition of 1×616, to 1439. A similar case occurs in finding the last remainder. But for this "doubtful case," which is peculiar and inherent to it, this method would be preferable to ordinary long division.

Divide 123456789 by 499.

```
          5..)123456789(247408
               2..
2×1......      2
              2365
               3..
4×1......       4
               3696
                1..
7×1......        7
                2037
                 0..
4×1......         4
                 4189
                  1..
8×1......          8
                  197 over.
```

```
499)123456789(247408
       2365
        3696
         2037
          4189
           197
```

This mode of division is called SYNTHETIC DIVISION, from *synthesis*, which means putting together. The ordinary mode might, per con-

tra, be called "Analytic Division." The 5 is called the SYNTHETIC DIVISOR.

Examination of this synthetic division shews that to find the true successive remainders, each figure of the quotient is added to the dividend two places to the right of that figure from which it was obtained. All this can be done mentally thus:

$$5..)\,123456789$$
$$247\cancel{3}08 \text{ and } 197 \text{ over.}$$
$$4$$

Wording: 5 in 12, 2′ (beneath 4), carry 2; in 23, 4′ (beneath 5), carry 3; in (34+2) 36, 7′, carry 1; in (15+4) 19, 3′, carry 4; in (46+7) 53, 10′ (altering the previous 3 into 4′), carry 3; in (37+4) 41, 8′, and (189+08) 197 over.

Ans. 247408 and 197 over.

or we may find decimal places:

$$5..)\,123456789$$
$$247\cancel{3}08\cdot3947895791583, \text{ \&c.}$$
$$4$$

Wording (continued): In 41, 8′, carry 1; in (18+0) 18, 3′, carry 3; in (39+8) 47, 9′, carry 2; in (20+3) 23, 4′, carry 3; in 39, 7′, carry 4; in 44, 8′, &c.

§ 6. When the synthetic divisor is 1, the doubtful case occurs so often that the following form will be found more convenient.

Divide 27689954372 by 999.

Dividing by 1000 we obtain for quotient 27689954 and 372 over; hence dividing by 999 we obtain the same quotient and 372 + 27689954 over, which being again divided by 999 will yield 27689 and 954 + 27689 over; and so on.

Mod. op.:

27689954	372
27689	954
27	689
	27
27717672	042
	2
	·044

Ans. $27717672\cdot\dot{0}4\dot{4}$.

The 2 carried from the sum of the remainders to the quotients being 2000 yields another complementary 2 to the remainder, which is accordingly 44. And $\frac{44}{999} = \cdot\dot{0}4\dot{4}$

§ 7. A slight extension of this method will adapt it to division by synthetic divisor 1, when A. C. is any number by which we can multiply mentally in one line. (Part I. Ch. IX. §§ 3, 4, 5, 6.)

Divide 27689954372 by 992.

Here the compensation is each time 8 times the quotient.

```
27689954 372
  221519|632
    1772|152
      14|176
        |112
--------|---
27913260|444
        |  8
         ---
         452   Ans. 27913260 and 452 over.
```

To decimalize $\frac{452}{992}$.

```
1...) 4|520
        32
      -----
       5|520
         40
       -----
        5|600
          40
        -----
         6|40, &c.
```

which can be worked nearly mentally:

```
452
  52
   60
    40
     48
     512
      160
       608
        128
         288
          896
          9024*
            320
             224       Ans. ·45564516129032..
```

* Here carriage makes the 8 a 9, and consequently we have to *add in* one more 8 to the 24, making the remainder 32.

173648146 ÷ 9983. (A. C. 17.)

```
17364|8146
   29|5188
     | 493
-----|----
17394|3827
     |  17
1....)3844
        91
      5046
        5450
           85
           918
            265   Ans. 17394·38505492.
```

Multiplication by 17 is done in one operation by the method of Part I. Ch. IX. § 3, and the same *mod. op.* can be used in all cases to which §§ 4, 5, 6, in the same chapter can be applied to the A. C.

327568994 ÷ 9799. (A. C. 201.)

```
32756|8994
  658|3956
   13|2258
     |2613
-----|----
33428|7821
     | 201
1....)8022
       1828
         481
         6418
          5386
           4865
            9454, &c.      Ans. 33428·8186549...
```

§ 8. All numbers not divisible by 2 or 5 can be multiplied so as to yield a product of the form 9999...... (Fermat's Theorem, p. 160).

Take 37. To obtain 9 in the units' place we multiply by 7.

$$\begin{array}{r} 7 \times 37 = 259 \\ 20 \times 37 = \underline{740} \\ 999 \end{array}$$

To change the tens' figure 5 into 9 we have to add 4 tens, and must therefore multiply by 2 tens; hence the required multiplier is 27. If, then, we have to divide by 37, we multiply the dividend by 27 and divide synthetically by (1...).

Take 67. Multiplier 597 obtained from the right, figure by figure.

```
  597
   67
  ---
  469
 649
399
```

Hence $67 \times 597 = 39999$, synthetic divisor 4...., without requiring to carry out the whole process.

§ 9. Decimalize $\frac{3258692}{177}$. $177 = 3 \times 59$.

```
3 ) 3258692    by common division.
6.) 1086230·666... by synthetic divsion.
      18410 689265536, &c.
```

Decimalize $\frac{14}{183}$.

Mod. op.: Multiplier,

```
  53
 183
 ---
 549
969
```

$14 \times 53 : 9699$. (A.C. 301, S.D. 1....)

```
1....)·07420
         6307
          4876
           9964
          502349
             2650
              7102
               3127
```

Ans. ·07650273.

Decimalize $\frac{5}{19}$. (S. D. 2.)

```
2.) 5·
     ·263157894 | 7, &c.
```

The remaining figures can be obtained by complementing (Ch. VI. § 10), and it is most convenient to write the second half of the period under the first:

```
2.) 5·
     ·263157894 |
      736842105
```

We know that we have arrived at the complementing stage because the remainder 14 complements the dividend 5 with respect to the divisor 19. (Ch. VI. § 12.)

§ 10. SYNTHETIC DIVISION SUBTRACTIVE.

827 ÷ 71.

$$\begin{array}{l} 7.)827 \\ \quad\;\; 1164788732394\ldots \end{array}$$

Wording: 7 in 8, 1′, carry 1; in (12 − 1) 11, 1′, carry 4; in (47 − 1) 46, 6′, carry 4; in (40 − 6) 34, 4′, carry 6; in (60 − 4) 56, 8, 7′, carry 7; in (70 − 7) 63, 8′, carry 7, &c.

§ 11. Synthetic division can of course be applied to other scales of notation by complementing to the next higher power of the radix.

13564 ÷ 266 (septenary).

$$\begin{array}{l} 3..)13564 \\ \qquad\;\; 34325541604,\ \&c. \end{array}$$

Wording: 3 in ten, 3′, carry 1; in (7 + 5) 12, 4′; in (6 + 3) 9, 3′; in 8, 2′, carry 2; in (2 × 7 + 3) 17, 5′, carry 2, &c.

EXERCISE LXX.

Simplify:

(1) $\{3 \times (49993 + 2 \times 3997)\} + 9998$.

(2) $\{9 \times (49993 - 2 \times 3997)\} \times 9998 \times 701$.

(3) $793{\cdot}\dot{7}1\dot{8} \times 3{\cdot}997$.

(4) ${\cdot}\dot{1}5384\dot{6} \times 3{\cdot}9$.

(5) ${\cdot}\dot{3}8461\dot{5} \times 91000$.

(6) By synthetic division (integral):

a. 7473684 ÷ 19, 29, 39, 49, 59.

b. 226543817 ÷ 199, 4999, 399, 3990.

c. 8543764333 ÷ 99, 999, 9999.

d. 4623814 ÷ 98, 997, 9996.

e. 54376146 ÷ 983, 9799, 9898.

(7) Decimalize by synthetic division, completing the period: $\frac{319}{41}$, $\frac{5284}{13}$, $\frac{5284}{17}$, $\frac{5284}{19}$, $\frac{5284}{23}$, $\frac{5284}{31}$, $\frac{5284}{47}$, $\frac{5284}{495}$, $\frac{5284}{2997}$.

(8) To 12 places: 73 ÷ 9995; 47 ÷ 989; 119 ÷ 9992.

§ 12. These methods will save much time to those who have acquired aptitude in detecting opportunities.

CHAPTER XIV.

MISCELLANEOUS EXAMPLES.

(Answers to money sums to be brought to the nearest farthing; other problems to three places, unless otherwise specified.)

(1) By vulgar fractions and by decimals, work the following, and shew that the results coincide:

a. $8\frac{7}{20} + 3\frac{11}{25} + 9\frac{4}{5} + 10\frac{3}{8} + 4\frac{15}{32}$.

b. $6\frac{2}{5} - 3\frac{15}{32}$.

c. $\frac{3}{8} \times \frac{1}{10} \times 2\frac{1}{2} \times \cdot 001$; $\frac{5}{16} \times \frac{1}{8} \times \frac{1}{25} \times 1000000$.

d. $14 \div 3\frac{1}{5}$; $2\frac{1}{2} \div \frac{8}{25}$; $\frac{1}{4} \div \frac{2}{625}$.

(2) Find the limits of the following:

$\cdot\dot{1}3\dot{5}$, $\cdot\dot{1}5\dot{3}$, $\cdot0\dot{1}3\dot{5}$, $\cdot2\dot{1}3\dot{5}$, $\cdot\dot{0}13\dot{5}$, $\cdot01\dot{3}\dot{5}$, $\cdot0\dot{1}5\dot{3}$.

(3) By decimal calculations only, find the following:

a. The cost of $5437\frac{5}{12}$ articles at £1. 13*s.* $10\frac{1}{4}d.$ each.

b. „ 287694 articles at $9\frac{5}{16}d.$ each.

c. „ 157 tons, 13 cwts., 2 qrs., 13 lbs., at £38. 10*s.* 8*d.* per ton.

d. The dividend on £347. 18*s.* 10*d.*, at 13*s.* $9\frac{7}{8}d.$ in the £.

e. The profit on £468. 17*s.* 5*d.*, at £9. 13*s.* $4\frac{1}{4}d.$ per cent.

f. The brokerage on £1267. 10*s.*, at $2\frac{1}{2}$ per mille.

g. The premium on £768, at $3\frac{1}{8}$ per cent., insured so as to recover both goods and premium in case of loss.

(4) Extract the square root of 191810·713444.

(5) Find to 4 places the difference between the square and cube roots of 32·14.

(6) Find to 3 places $2 \times \sqrt{3} - \frac{1}{2} \times \sqrt{12} + \sqrt{27}$.

(7) Prove that $\sqrt{18\cdot\dot{7}} = 4\frac{1}{3}$.

(8) Find the discount, the simple interest, and the compound interest, on £465 for 18 months at $4\frac{1}{2}$% (compound interest calculated half-yearly).

(9) Find a fourth proportional to $1\frac{1}{2}$, ·09, $\frac{9}{20}$.

(10) Shew that $\frac{1}{2600}$ of £10. 16*s.* 8*d.* is equal to ·002 of £2. 1*s.* 8*d.*, and that ·001 of £2 $= \frac{1}{2{\cdot}08\dot{3}}$ of 1*d.*

(11) Reduce to a decimal: $\dfrac{3}{4+\dfrac{2}{3+\dfrac{2}{3}}}$

(12) Extract the square root of 272·316004.

(13) Extract to 9 places the square root of ·034.

(14) Sum the series $1 + \frac{3}{2} + 2 + \frac{5}{2} + \ldots$ to 30 terms.

(15) Sum the series $\frac{387}{1000} + \frac{387}{10000000} + \frac{387}{10000000000}$ to infinity.

(16) A can dig a certain ditch in 3 days, B in 4 days, and C in 5 days. How long will it take the three together to dig the ditch, and what fraction of it is dug by each?

(17) Find the amount at compound interest at $12\frac{1}{2}$ % on £819. 4*s.* for 6 years, reckoned yearly.

(18) Extract to 5 places the cube root of $\cdot 0\dot{3}\dot{4}$.

(19) If the carriage of 1 cwt., 12 lbs. for 105 miles comes to 3*s.* $10\frac{1}{2}$*d.*, what should be charged for the carriage of 8 cwt., 1 qr., 24 lbs. for 245 miles?

(20) Simplify $\dfrac{5\frac{2}{3}}{7\frac{1}{2}}$ of $\dfrac{21{\cdot}25}{{\cdot}046875}$.

(21) Find in what time £452. 10*s.* will amount to £644. 16*s.* 3*d.* at $4\frac{1}{4}$ % per annum simple interest.

(22) Express £4. 6*s.* $4\frac{3}{4}$*d.* + $\frac{1}{5}$ of 1 farthing as a decimal of £5.

(23) A grocer mixes 3 cwt., 15 lbs. of sugar at $5\frac{1}{2}$*d.* per lb. with 10 cwt., 10 lbs. at 4*d.* per lb. At what price per lb. should he sell the mixture to gain 25 %?

(24) Find G.C.M. and L.C.M. of 6·3375, 73·125, 39, 12·1875.

(25) Find the cost of 6 reams, 13 quires, 10 sheets, at £1 per ream.

(26) A man who has $\cdot\dot{3}$ of the profits of a partnership sells ·25 of his share, and the buyer makes £89. 6*s.* 8*d.* per annum. What *is the* yearly income of the whole business, and if the buyer pay *£1000* for his share, what interest does he get for his money?

(27) Sum the series:

a. $2 + 5 + 8 + 11 +$ to 20 terms.

b. $3 + \frac{3}{10} + \frac{3}{10^2} +$ to 10 terms.

(28) One sample of tea costs 3*s.* 4*d.* per lb., and another 4*s.* per lb. At how much per lb. must the mixture of 60 lbs. of the former and 25 lbs. of the latter be sold to gain $7\frac{1}{2}\%$?

(29) Find the side of a square grass plot which is of the same area as a rectangular grass plot 63 ft. long and 28 ft. broad.

(30) Find the difference between the simple and compound interest reckoned yearly, on £210 for 2 years at 5 %.

(31) Find the difference to 3 places between $\sqrt[3]{3}$ and $\sqrt{2}$.

(32) A carpenter makes 2 chairs in 3 days, and 3 chairs and 1 table in 8 days. In what time would he make 6 chairs and 3 tables?

(33) A man walks 80 miles; he begins by walking 4 hours a day, at the rate of 5 miles an hour, and each day increasing the number of hours by 1, he diminishes the pace by 1 mile per hour. How many hours does he walk, and how many days?

(34) If a model of a cathedral is to be made on the scale of 6 ft. to $\frac{3}{4}$ of an inch, what will be the dimensions in the model of a tower 120 ft. high, a roof 50 ft. long, and a floor 40 square yards in area?

(35) Simplify:

a. $\left\{\frac{3}{5} \div \left(\frac{4}{25} - \frac{1}{9}\right)\right\} + \left\{1 \div \left(\frac{2}{5} - \frac{1}{3}\right)\right\} - \left\{1 \div \left(\frac{2}{5} + \frac{1}{3}\right)\right\}$

b. $\left(\frac{2}{3} + \dfrac{\frac{4}{5} - \frac{2}{3}}{1 + \frac{8}{15}}\right) \times \frac{5}{6} \div \left(1 - \frac{2}{3} \times \dfrac{\frac{4}{5} - \frac{2}{3}}{1 + \frac{8}{15}}\right)$

c. $\dfrac{\frac{2}{3} \times \sqrt{3} \times \frac{1}{2} \times \sqrt[3]{2}}{\frac{5}{8} \times \sqrt[3]{2} \times \frac{2}{7} \times \sqrt{3}}$

(36) The items of a journey on the continent are £4. 10*s.* 6*d.*, 50·75 francs, 66 thalers. Find the whole cost of the journey in English, French and German money, when £1 = 25·22 francs = $6\frac{2}{3}$ thalers.

(37) In the morning I solved $\frac{3}{4}$ of $\frac{1}{3}$ of a certain number of problems; by the end of the afternoon I had done $\frac{2}{5}$ of $\frac{2}{3}$ of the number. Suppose the whole number of problems to be 300; how many did I solve during the afternoon?

(38) Multiply and divide to 4 places $\cdot\dot{3}8095\dot{2}$ by $1\cdot\dot{3}$, and verify the results by reducing the decimals to their limits.

(39) How much 3 per cent. stock must I sell out to pay a debt of £550, the price of stock being $94\frac{1}{8}$, and brokerage $\frac{1}{8}$ %?

(40) Find accurately the value of:

a. £$\cdot 003$, £$\cdot 0\dot{0}\dot{3}$, £$\cdot\dot{0}0\dot{3}$, and £$\cdot 00\dot{3}$.

b. $\cdot 08\dot{3}$, $\cdot 0\dot{8}\dot{3}$, $\cdot\dot{0}8\dot{3}$ of a ton.

c. $\cdot 41\dot{6}$, $\cdot\dot{4}1\dot{6}$ of 1 lb. troy.

d. $\cdot 0099$, $\cdot 00\dot{9}$ of 1 lb. av.

e. $\cdot\dot{1}0\dot{8}$, $\cdot 11\dot{8}$ of 1 gallon.

(41) Find the reciprocal of the difference between $31\cdot 24$ and $31\cdot 23768\dot{1}4285\dot{7}$.

(42) Which is cheaper, to buy napoleons (20 fr.) at 15*s.* $10\frac{1}{2}$*d.*, or $25\cdot 22$ fr. for a pound sterling.

(43) A kilogramme is $2\cdot 205$ lbs. avoirdupois; a French ton = 1000 kilogs. What fraction of an English ton is a French ton?

(44) Find the difference between $5\frac{1}{2}$ sq. ft. and $5\frac{1}{2}$ ft. square.

(45) How much stock must I sell out of the Consols when they are at $93\frac{3}{4}$ (brokerage $\frac{1}{8}$), to raise a sum of £1265?

(46) Find the square roots (to 4 places of decimals) of 1979649, $7\frac{1}{8}$, $\frac{3}{7}$, $\cdot 25$, $\cdot 35$, $\cdot 000025$, $\cdot 025$, $\cdot 001$.

(47) Find the cost of a case lined with tin, 5 ft., 10 in. long, 4 ft. broad, and 1 ft., 8 in. deep, inside measurement, at 2*d.* per sq. ft. for the wood, and $2\frac{1}{4}$*d.* per sq. ft. for the tin.

(48) If a napoleon be worth 15*s.* $10\frac{1}{2}$*d.*, find the lowest exact number of napoleons that must be given for an exact number of English sovereigns, stating the number of each.

(49) Simplify to 5 places of decimals:

$$\sqrt{1024} + \sqrt{102\cdot 4} + \sqrt{10\cdot 24} + \sqrt{1\cdot 024} + \sqrt{\cdot 001024}.$$

(50) Find the cost of papering a room 16 ft. long, 11 ft. wide, and 10 ft. high, with paper 30 inches broad, at $7\frac{1}{2}$*d.* a yard.

(51) Multiply (by the method of duodecimals) 7 ft., 5 in., 8 parts, by 9 ft., 4 in., 11 parts.

(52) Express the result of the last question as square inches.

(53) Arrange the following journeys in order of rapidity:

Name of Line.	Departure from London.	Arrival at		Distance.
Great Northern	10 a.m.	Aberdeen	3 a.m.	526 m.
Great Western	9.15 a.m.	Penzance	9.30 p.m.	328 m.
North Western	10 a.m.	Carlisle	6.10 p.m.	299 m.
Great Eastern	5 p.m.	Yarmouth	9.45 p.m.	146 m.
South Western	11.10 a.m.	Weymouth	4 p.m.	147 m.

(54) The fares for these journeys are £4, £3. 10*s.*, £2. 15*s.*, £1. 10*s.*, £1. 9*s.* 6*d.* respectively. Arrange them in the order of cheapness.

(55) What is the average charge per mile?

(56) If I invest 7000 guineas in the $3\frac{1}{2}$ per cents. at 93, what is my nett income, deducting 5 % income-tax?

(57) A room costs £8 to paper. What would a room cost half as high again, half as long again, and half as broad again, with paper costing half what it did before per yd.?

(58) If 144 men can dig a trench 40 yds. long, $1\frac{1}{2}$ ft. broad, and 48 ft. deep, in 3 days of 10 hours each, how long must another trench, 5 ft. deep and $2\frac{1}{4}$ ft. broad, be, in order that 51 men may dig it in 15 days of 9 hours each?

(59) The discount on a sum due one year hence at 5 % per annum is £15. What is the sum?

(60) How many dice a third of an inch long can be packed into a box whose dimensions inside are 2, 3 and 4 ft. respectively?

(61) How many dice a third of an inch long can be packed into a cubical box $1\frac{1}{2}$ ft. long inside?

(62) Find the square root of $\cdot\dot{3}0864197\dot{5}$.

(63) A mass of lead ore weighing 800 grains troy, was found to contain ·6 grain of silver. What is the value of silver in one ton of the ore, at the rate of 5*s.* the oz. troy?

(64) If £24. 7*s.* $10\frac{1}{2}$*d.* be paid as income-tax on an income of £650. 10*s.*, what ought to be paid at the same rate on an income of £2450. 6*s.* 8*d.*? And at what rate in the £ is the tax levied?

(65) In how many years will £625. 10*s.* amount to £813. 3*s.*, at 4 % simple interest?

(66) Find the value of 159 cwt., 3 qrs., 22 lbs., at £2. 12*s.* 6*d.* per cwt.

(67) Add $\frac{1}{8}$ of a guinea, £$\frac{9}{32}$, $\frac{1}{20}$ of a crown, and $\frac{15}{16}$ of a shilling, and express the whole as a decimal of £1.

(68) What sum of money will at $4\frac{1}{4}$ % simple interest amount in $3\frac{1}{2}$ years to £1497. 4*s.* 1*d.*?

(69) If the price of 3 bushels of wheat is 16*s.* 9*d.*, find the price of 12 qrs., 2 bus., 1 peck.

(70) Find the cost of making a road, length 9 miles, 5 fur., 44 yds., at £25. 8*s.* 4*d.* per mile.

(71) In what time will a sum of money double itself at 5 %, (*a*) at simple interest, (*b*) at compound interest, reckoned yearly?

(72) If when wheat is 60*s.* a quarter the sixpenny loaf weigh 4 lbs., how much should be paid for 25 lbs. of bread when wheat is 40*s.* a quarter?

(73) What decimal of an English mile is an Indian league, of which 30 go to a degree (60 geographical miles), 1 geographical mile being 1·1508 English miles?

(74) A Russian gold ducat is worth 9*s.* 5*d.* sterling; 3 roubles make a ducat, and 100 kopeks 1 rouble. Find in English money the value of 315 ducats, 2 roubles, 80 kopeks.

(75) Gun-metal consists of 100 parts of copper and 11 of tin. How much of each metal will there be in a cannon weighing 3 tons, 9 cwt., 1 qr., 16 lbs.?

(76) Find the income arising from investing £740 in the 3 per cents. at $92\frac{1}{2}$.

(77) Reduce to lowest terms $\frac{403}{899}$, $\frac{5371}{59736}$.

(78) Find the sum of 19 terms of the series 7, 14, 21, &c

(79) If a degree of longitude at X—— be $\frac{2}{3}$ of the length of a degree at the equator, how many miles at the latitude of X—— will the sun pass over in a minute and a half, given that the equator is 131470565 feet long?

(80) A ship's company take a prize of £1000, which is to be divided amongst them in proportion to their pay and to their time of service; the officers, 4 in number, have 40*s.* each a month, and the midshipmen, 12 in number, have 30*s.* each a month, and they have all served six months; the sailors, 110 in number, have each 22*s.* a month, and have served 3 months. Find each man's share.

(81) An Austrian bankrupt owes a London merchant 5784 florins, and pays $68\frac{1}{2}$ %. How much is that in the £, and how much sterling money will be remitted to the Englishman (exchange 11·45 fl. = £1)?

(82) The rent-roll of a certain estate amounts to £3580 a-year. The repairs average $7\frac{1}{2}$ % per annum. Find the value of the estate at 28 years' purchase.

(83) Which is the heavier income-tax, 3 % or 7*d.* in the £?

(84) A steamer working with a given force can travel down the river at the rate of $12\frac{1}{2}$ miles an hour. Of this speed, $\frac{2}{7}$ is due to the current. How long would the steamer take to travel 15 miles up the stream?

(85) Find the rent of 204 acres, 1 rood, 20 poles, at £2. 15*s.* 9*d.* per acre.

(86) Find the *discount* on £972, due 10 months hence, at $5\frac{1}{4}$ % per annum, and shew what rate of *interest* is charged in this case.

(87) If the rations of 3264 men for 48 days cost £4787. 4*s.*, what is the cost of the rations of 5000 men for 90 days?

(88) Two persons have invested £11. 17*s.* $2\frac{3}{4}$*d.* and £17. 16*s.* $8\frac{1}{2}$*d.*, and the return is £46. 2*s.* $0\frac{3}{4}$*d.* Find within a farthing what the share of each must be.

(89) Of £121. 13*s.* $4\frac{3}{4}$*d.* and £29. 8*s.* 10*d.*, what percentage is each of the other (5 places)?

(90) Determine without any superfluous work $\sqrt{1{\cdot}0097626}$ to 8 places.

(91) Find correct to one 10,000th of a unit $16{\cdot}112734 \times {\cdot}20708 \times 1146{\cdot}339 \div {\cdot}00007$.

(92) A ship valued at £14,500 is insured at £3. 10*s.* %, and her cargo valued at £32,000 is insured at £4. 17*s.* 6*d.* %. Find the whole cost of insurance.

(93) I invested £680 in Consols at $89\frac{3}{8}$; 3 days later the funds rose to $90\frac{7}{8}$. What would have been my loss of income had I waited these 3 days, brokerage $\frac{1}{8}$ %?

(94) A speculator bought in Consols at $88\frac{3}{8}$, and sold out at $91\frac{7}{8}$, brokerage $\frac{1}{8}$ % each time; his total gains amounted to £350. Find the value of the stock when bought in.

(95) I bought $3\frac{1}{2}$ per cent. stock at $95\frac{1}{2}$, and after drawing one half-yearly dividend, I sold at $92\frac{7}{8}$; my total loss of capital amounted to £8. 15*s*. Find the amount of stock I had held.

(96) The prices of the 3 per cent. Consols, and Midland Railway Stock, paying $5\frac{1}{4}$ %, were quoted at $95\frac{3}{8}$ and $108\frac{1}{2}$ respectively. Find the difference in income from investing £100 in each.

(97) Find the average of $17\frac{1}{2}$, $25\frac{1}{4}$, $96\frac{3}{8}$, 10, 0, $42\frac{3}{4}$, 56, and express the answer decimally.

(98) The income of a parish is £6529. 10*s*. 6*d*. How much in the £ will produce a rate of £150?

(99) If the time after 1 p.m. is $\frac{7}{15}$ of the time before midnight, what o'clock is it?

(100) Find cube root of $1776\frac{325}{729}$.

(101) Express 1 acre, 3 roods, 26 perches, as the decimal (in full) of a square mile.

(102) If 120 men make an embankment $\frac{3}{4}$ of a mile long, 30 yards wide, and 7 yds. high, in 42 days, how many men would it take to make an embankment 1000 yds. long, 36 yds. wide, and 22 ft. high, in 30 days?

(103) A person invests £1365 in the 3 per cents. at 91; he sells out £1000 stock when they have risen to $93\frac{1}{2}$, and the remainder when they have fallen to 85. Find his gain or loss.

(104) A and B have gained £600 between them; A has to receive 10 % less than B. Find their respective shares.

(105) Distribute £1250 among A, B and C, giving to A 15 % more, and to B 12 % less than to C.

(106) Distribute £760 among A, B and C, giving to A $17\frac{1}{2}$ % less than B's share, which is 20 % more than C's share.

(107) A bankrupt's estate amounts to £910. 3*s*. $1\frac{1}{2}$*d*. and his debts to £1875. What can he pay in the £, and what will a creditor lose on a debt of £57 ?

(108) An estate with a rental of £8790 is sold for £351,600 In order that it may yield the purchaser $3\frac{3}{4}$ % for his money, how much % must he raise the rent ?

(109) A square court-yard costs £38. 10*s*. 5*d*. to pave, at 3*s*. 9*d*. per square yard. Find the length of its side.

(110) Express 37048 (decimal) in the nonary scale; also 347102 (nonary) in the decimal scale.

(111) The Hanoverian mile is 25400 Hanoverian feet long, each foot being ·9542 English feet. Find to 4 places of decimals the fraction that an English is of a Hanoverian mile.

(112) How many times will a wheel whose diameter is $3\frac{2}{7}$ feet revolve in travelling over 5 miles ? (N.B. Circumference : diameter = 3·14159 : 1.)

(113) If a package weighing 7·5 cwts. be carried 125 miles for 14*s*. 7*d*., how much will be charged for the carriage of 3 tons, 15 cwts. for a distance of 200 miles ?

(114) If 770 gallons of creosote at 1*d*. per gallon have the heating power of 8·75 tons of coal at £·641$\dot{6}$ per ton, find the yearly saving in money in a factory which burns 1000 gallons a day, omitting 52 Sundays.

(115) Extract the square root of 1194·3936 and of $\frac{14 \cdot 4}{16 \cdot 9}$.

(116) Simplify :

$$\frac{\frac{5}{8} \times \frac{1}{1\frac{1}{4}} + 1\frac{5}{7} \text{ of } \frac{1}{\frac{4}{7}}}{8} + \frac{\frac{21}{3\frac{1}{3}} \times \frac{5}{18\frac{3}{4}} \times 37\frac{1}{2}}{\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{2}\times\frac{1}{5}}} - \left(\tfrac{1}{7} \text{ of } \frac{25}{3\frac{1}{8}} \text{ of } \frac{3\frac{1}{3}}{6\frac{2}{3}} + \tfrac{1}{14} \text{ of } \frac{1}{1\frac{1}{3}}\right)$$

(117) Express + 40° and – 40° Fahrenheit on the Centigrade and Réaumur scales.

(118) Find the cost of $3047\frac{5}{11}$ articles at £1. 15*s*. $9\frac{1}{2}$*d*. each.

(119) What capital will at $3\frac{1}{2}$ % in three months amount to £348. 0*s*. $4\frac{1}{2}$*d*. ?

(120) A bankrupt's debts amount to £27485. 10*s.* 9*d.* and his assets are worth £9328. 6*s.* 3*d.* What is he to pay on each of the following debts: £248. 14*s.* 5*d.*; £7642. 10*s.* 6*d.*; £19. 4*s.* 2*d.*?

(121) Find the cost of 47 iron plates, each weighing 127 tons, 13 cwt., 1 qr., 19 lbs., at £50. 6*s.* 10*d.* per ton.

(122) Find the value of 21 acres, 2 roods, 15 perches, at £37. 15*s.* 6*d.* per acre.

(123) If the sixpenny loaf weigh 4·35 lbs. when wheat is at 5·75 shillings per bushel, what weight of bread ought to be purchased for 18·13 shillings when wheat is at 18·4 shillings per bushel?

(124) Three gardeners working full time can plant a field in 10 days. How long will it take them if one of them works half time?

(125) A person bought into the 3 per cents. at 98, and after receiving 3 years' interest sold out at 90. How much % on the sum invested did he gain or lose?

(126) A Turkey carpet measuring 12 ft., 6 in., by 11 ft., 6 in., is laid down on the floor of a room measuring 14 ft. by 13 ft. Determine the quantity of floorcloth necessary to complete the covering of the floor, and its price, at 4*s.* per square yard.

(127) Reduce 9*s.* $11\frac{3}{4}$*d.* to the fraction of half-a-sovereign.

(128) Simplify $\frac{1{\cdot}18}{{\cdot}152} \times \frac{3{\cdot}04}{2{\cdot}95} \div {\cdot}00125$.

(129) Find the cost of 457 tons, 13 cwt., 3 qrs., 19 lbs., at £5. 17*s.* $8\frac{1}{2}$*d.* per ton. Also of 17 lbs., 9 oz., 17 dwts. troy, at £3. 15*s.* $10\frac{1}{4}$*d.* per oz.

(130) Find the compound interest for £55 for 1 year, reckoned quarterly, at 5 % per annum.

(131) Find, without unnecessary work, to 3 places:

a. ${\cdot}400\dot{8}\dot{6} \times 16{\cdot}\dot{0}5\dot{9} \times 2618{\cdot}0\dot{8}5\dot{3} \times {\cdot}00035$.

b. ${\cdot}41\dot{9} \times 9{\cdot}\dot{8} \times 720 \times 43{\cdot}\dot{1}5\dot{6}$.

c. $3{\cdot}1415926536 \times \sqrt{7000} \times \sqrt[3]{\frac{1}{8}} \times 1870$.

(132) Find the difference between the square and cube roots of 3915380329 (to 1 place).

(133) Simplify $\sqrt{25\frac{44759}{117649}} - \sqrt[3]{25\frac{44759}{117649}}$. Also find the value of $\sqrt[3]{{\cdot}000729} - \sqrt{{\cdot}000729}$; and explain why the sign $\sqrt{\ }$ is placed first in one case, and $\sqrt[3]{\ }$ in the other.

(134) A buys 134 gallons of beer for £11. 18*s.*; 6 gallons are lost by leakage; he sells the rest in jugs, holding $\frac{3}{8}$ of a quart, at $2\frac{1}{4}d.$ per jug. Find his total profit, and his profit %.

(135) The receipts of a railway company are apportioned in the following manner: 49 % for working expenses, 10 % for the reserve fund, a guaranteed dividend of 5 % on one-fifth of the capital, and the remainder, £40000, for division among the holders of the rest of the stock, being a dividend of 4 % per annum. Find the capital and the receipts.

(136) Find the square root of 24·2064; of 3124·81; of $2{\cdot}42064 \times 312{\cdot}481$ and of $\frac{2{\cdot}42064}{312{\cdot}481}$.

(137) Find the cost of paving a hall 50 yds. long by 50 ft. broad, with marble slabs 1 ft. long and 9 in. broad, the price of the slabs being £5 per dozen.

(138) Simplify $\frac{{\cdot}0075 \times 2{\cdot}1}{{\cdot}0175} + \frac{4{\cdot}255 \times {\cdot}0064}{{\cdot}00032}$.

(139) Find the square root to 8 places of decimals of 38715, 1500, 150, 7, ·7, ·07, 49, 4·9, ·49, $\cdot 4\dot{9}$, $\cdot 3402\dot{7}$.

(140) The quadrant is divided into 90^0 and also into 100 grades. Express $37\frac{1}{3}$ deg. + $50\frac{2}{9}$ gr. both as degrees and as grades.

(141) A person having £1000 invests in the 3 per cents. at $92\frac{3}{8}$, brokerage $\frac{1}{8}$; after 3 years he sells at $94\frac{5}{8}$, and again pays $\frac{1}{8}$. What did he receive as interest, and what did he gain on the whole?

(142) Divide £26. 3*s.* 3*d.* between 3 persons so that their shares may be in the proportion of £2. 18*s.* 6*d.*, £1. 19*s.* and £1. 9*s.* 3*d.*

(143) If the price of candles $8\frac{1}{2}$ inches long be 9*d.* per half-dozen, and that of candles of the same thickness and quality $10\frac{1}{4}$ inches long be 1*s.* $4\frac{1}{2}d.$ for 9 candles, which is the cheaper kind, and how much % is lost by buying the dearer?

(144) Find the profit or loss per dozen on a quantity of wine "laid down" in 1848 at a cost of £2. 8*s.* per dozen, and sold in 1870 at £5. 5*s.* per dozen, reckoning compound interest annually at 5 %.

(145) Express $\cdot\dot{3}$ (in the quinary scale) as a decimal fraction.

(146) I sold goods at a loss of $7\frac{1}{2}$ %; had I sold them at a gain of $7\frac{1}{2}$ % I should have realized £3. 15*s.* more than I actually received. Find the cost of the goods.

(147) Multiply by duodecimals 9 ft., 7 in., 3 pts., by 5 ft., 7 in., 11 pts., and the product by 2 ft., 7 in. What does the product become when expressed in cubic feet and inches?

(148) In 1841 the population of Great Britain was 21,476,000, and that of Ireland 7,310,000; in 1851 the former had increased 8·45 % and the latter had decreased 12·5 %. Find the increase % in the population of the United Kingdom.

(149) Reduce $\frac{68}{157}$ to a converging fraction; give the several convergents and the limit of error in each.

(150) A certain book costs in production 2*s.* $4\frac{1}{2}$*d.* per copy, and its retail price is 7*s.* 6*d.*; the publisher allows the bookseller 25 % on the retail price, and gives 13 copies to the dozen; 3900 copies are printed and sold; the author is to have half the profits. How much will he receive?

(151) A tenant pays a corn rent of 20 quarters of wheat and 12 of barley, Winchester measure. What is the value of his rent, wheat being at 60*s.* and barley at 54*s.* a quarter imperial measure, reckoning a Winchester bushel to be $\frac{32}{33}$ of an imperial bushel.

(152) Express as a decimal $\frac{117}{5^{11}\times 2^7}$.

(153) A certain Building Society accepts £11. 14*s.* 1*d.* at the beginning of the year in lieu of 12 monthly instalments of £1 each. What yearly payment will at this rate discharge a monthly liability of £5. 17*s.* 10*d.*?

(154) I wish to borrow of a Building Society £600, to be paid off in 15 years by monthly instalments, paying interest at 6 % on the whole sum borrowed for the whole time. What should be my *yearly* payment at £11. 14*s.* 1*d.* instead of £1 per month?

(155) I hold £43. 17*s.* 5*d.* Building Society stock for 7 months, and the year's balance sheet shews a dividend of £11. 13*s.* $8\frac{3}{4}$*d.* per cent. per annum. What dividend should I receive?

(156) A cistern holding 820 gallons is filled in 20 minutes by 3 pipes, the first of which conveys per minute 10 gallons more, and the second 5 gallons less, than the third. How much flows through each pipe per minute?

(157) Gold is 19·3 times, and copper is 8·62 times as heavy as water. How many times as heavy as water is standard gold, which is a mixture of 11 parts of gold and 1 of copper?

(158) The annual retreat of the equinox along the ecliptic is 50″. In what time will the equinox be carried round the whole circle of the ecliptic (360^0)?

(159) In what time will the sun move through 50·1″, when it traverses 360^0 in 365 days, 6 hours, 9 min., 9·6 sec., the motion being supposed uniform?

(160) A bankrupt's stock was sold for £520. 10*s.*, at a loss of 17 % on the cost price. Had it been sold in the course of trade, it would have realized a profit of 20 %. How much was it sold below the trade price?

(161) A foreign Government contracts for three loans in different markets: the first, a 5 % loan for 20 millions; the second, a 4 % loan for 12 millions; the third, a $3\frac{1}{2}$ % loan for 10 millions. For the first the Government received £65, for the second £50, for the third £42 for every £100 stock. How much money does Government receive for all these loans, what average rate of interest is paid on the money actually received, and on which of the three loans does the Government pay the lowest rate of interest?

(162) A besieged garrison loses 5 men on the first day, 10 on the second, 15 on the third, and so on for 30 days, when the commandant, finding that he had only $\frac{1}{4}$ of his original garrison left, surrendered. How many men were there at first?

(163) A merchant sells tea to a tradesman at a profit of 60 %, but the tradesman becomes bankrupt and only pays 13*s.* 4*d.* in the £. How much per cent. does the merchant gain or lose?

(164) Find the sum which must be invested in the 3 per cents. at 90, to amount in $23\frac{1}{2}$ years to £3317 sterling, the price of the funds remaining unchanged. If we sold out at 96, how many years sooner could the required amount be realized?

(165) The sidereal year consists of 365 days, 6 hours, 9 minutes, 9·6 seconds, reckoned in mean solar time, or of 366 days, 6 hours, 9 minutes, 9·6 seconds, reckoned in sidereal time. Find the ratio of a sidereal to a solar day, to 5 places.

(166) Two shepherds, A and B, owning a flock of sheep, agree to divide it; A takes 144 and B 184 sheep, paying £70 to A. Find the value of 1 sheep.

(167) Among how many persons must £158. 17*s.* 3*d.* be divided, in order that half of them may have 10*s.* 7*d.* each, and the other half 7*s.* 2*d.* each?

(168) A and B have the same sum of money; A buys equal amounts of 3 per cent. stock at 91, and of $3\frac{1}{2}$ per cent. at $97\frac{1}{2}$; B invests his money equally in the purchase of the same stocks. A's income being 1*s.* more than B's, how much money had each?

(169) Assuming a cubic foot of water to weigh 1000 oz. av., find the weight of a rainfall of one inch over an acre of ground.

(170) A person has a sum invested in the 3 per cents., which he sells and invests in the $3\frac{1}{2}$ per cents. at $87\frac{1}{2}$. If his income remains the same, what was the price of the 3 per cents.?

(171) A takes 6 steps while B takes 7; but 4 of A's steps are equal to 5 of B's. Which is the quicker walker?

(172) An army lost 18 per cent. of its strength by disease and desertion, and then 14 per cent. of the remainder in battle; the number then remaining was 84,624. Of how many did it originally consist?

(173) A person sells £5000 Consols at $94\frac{7}{8}$, and on their rising he sells £5000 more at $95\frac{5}{8}$; on their again rising he buys back the whole £10000 at 96. What does he lose?

(174) If gold is 19·3 times as heavy as water, and copper is 8·96 times as heavy as water, how many times its own bulk of water will a crown weigh composed of 9 oz. of gold and 15 oz. of copper?

(175) Find by inspection (table, p. 152) the number of recurring and non-recurring figures in the decimalization of each of the following fractions: $\frac{1}{8}$, $\frac{1}{625}$, $\frac{1}{16\times25}$, $\frac{1}{2^{10}\times5^{10}}$, $\frac{1}{2^5\times5^3}$, $\frac{1}{7}$, $\frac{1}{56}$, $\frac{1}{91}$, $\frac{1}{21}$, $\frac{1}{7\times37}$, $\frac{1}{2\times5\times7\times37}$, $\frac{1}{13\times79}$, $\frac{1}{13\times41}$, $\frac{1}{19\times23}$, $\frac{1}{2\times3\times5\times7\times11\times13}$.

(176) If a mass of silver be worth £720,000 when silver is worth £4. 4s. per lb. av., how much would the mass be worth if silver fetched 13·75 shillings for 2·5 ounces troy?

(177) A, B and C begin playing with £1. 6s. each; A wins 5s. each game, and B loses $\frac{2}{15}$ of A's gains. After how many games will C have nothing left, and what will A then have?

(178) Reduce $\frac{222}{931}$ to a converging fraction, and give the several convergents with limits of error to each.

(179) I sold a watch for 5 guineas and thereby cleared 20 % of my money. How much % should I have gained or lost if I had sold it for $4\frac{1}{2}$ guineas?

(180) If the French 3 per cents. are at 60 when the English are at 95, the exchange between the countries being 25 fr. per £1, how much French stock in francs can be bought by selling £6000 out of the English funds?

(181) I bought silk at fr. 7·40 per metre (39·37 inches) and sold it for 6s. $10\frac{1}{2}d.$ per yard. Find (to 2 places) my profit or loss %, the rate of exchange being fr. 25·35 = £1.

(182) I had a cistern 5 ft., 7 in. long, 3 ft., 11 in. broad, 2 ft., $8\frac{1}{2}$ in. deep, re-lined with zinc at $8\frac{3}{4}d.$ per square foot; the plumber allowed me $\frac{7}{8}d.$ per square foot for the old zinc. How much had I to pay?

(183) Find the length of a cubical tank holding 1 ton of water, if a cubic foot of water weighs 1000 oz. av.

(184) Find in two ways $\sqrt[6]{1061520150601}$.

(185) Find in two ways to 3 places $\sqrt[6]{7 \cdot 358}$.

(186) If a merchantman sailing $9\frac{1}{2}$ knots an hour is chased by a gunboat steaming $10\frac{3}{4}$ knots, how far ahead must the sailing vessel be just to escape into port from which she is $15\frac{1}{2}$ knots at the commencement of the chase?

(187) Divide £14. 11s. $8\frac{1}{2}d.$ into two parts that shall have to one another the same ratio as the sum of $2\frac{5}{8}$ and $1\frac{4}{7}$ has to their difference.

(188) Also the same ratio as the product of the two numbers has to the quotient, the greater being divided by the less.

(189) Find the side of a square field an acre in extent (to tenths of a yard).

(190) The discount on a sum due 3 months hence at 5 % was £17. 10*s.* What is the sum?

(191) A grocer mixes 3 cwt. of tea at £16. 16*s.* per cwt. with 1 cwt. at £19. 12*s.* At what rate per lb. must he sell the mixture so as to gain 4 %?

(192) A person having invested a sum of money in the 3 per cent. Consols receives annually therefrom £233 after deducting the income-tax of 7*d.* in the £. How much stock does he hold, and how much will it be sold for, at $94\frac{1}{4}$, brokerage $\frac{1}{8}$.

(193) From 122·5 grains of chlorate of potash there can be obtained 48 grains of oxygen gas; 16 grains of oxygen occupy a space of 44·4 cubic inches. What volume of oxygen could be obtained from a ton av. of chlorate of potash?

(194) Find the length of the side of a cubical tank which contains 15 cwt., 7 lbs., 8 oz., of water, 1 cubic foot of which weighs 1000 oz.

(195) Which money sums will when decimalized yield recurring decimals? and how could you get rid of the recurring figure if required?

(196) How long will it take me to travel 5 Russian versts at the rate of $8\frac{1}{2}$ miles an hour?

(197) Find the sum to be awarded on £87. 13*s.* 10*d.* at £7. 15*s.* $9\frac{3}{8}$*d.* %.

(198) Express $\frac{224\cdot7}{365\cdot256}$ as a continued fraction, and find the five first convergents.

(199) After the outbreak of the Prusso-French war in 1870 the Prussian Government issued a 5 % war loan at 88; the French 3 per cents. stood at $65\frac{1}{2}$. State the ratio of the two rates of interest.

(200) If 9000 persons travelling each 20 miles a week pay a railroad company £900 in one week, how many persons travelling each 30 miles weekly will give a receipt of £62,400 a year when the charge for travelling per mile is reduced one half?

(201) Find the amount of the national debt from the following sums paid as annual interest:

£3 per cent. Consolidated Annuities		£11871403	10	0
£3 per cent. Reduced	ditto	3188376	11	7
New £3 per cent.	ditto	6633792	10	10
New £3. 10*s.* per cent.	ditto	8426	2	4
New £5 per cent.	ditto	21687	9	8
New £2. 10*s.* per cent.	ditto	96176	7	0
Interest on the Government Debt to the Bank of England at 3 %		330453	0	0
Ditto to the Bank of Ireland at 3 %		78923	1	6

(202) A bar of gold weighing 8·75943 kilogs., of which $\frac{19\cdot58}{24}$ is fine, is sent over from Paris, and sold here at £3. 17*s.* 9*d.* per standard oz., which is $\frac{22}{24}$ pure. How many francs must be remitted in payment, exchange being 25·35 fr.?

(203) Find the value of a bar of gold which weighs 11 lbs., 8 oz., 7 dwt., 12 grs., and is $\frac{21\frac{1}{2}}{24}$ pure at the rate of £3. 17*s.* 9*d.* per oz. standard.

(204) How many cubic yards of gravel will be required for a walk surrounding a rectangular lawn 200 yards long and 100 yards wide, the walk to be 3 yards wide, and the gravel 3 inches deep?

(205) Find (to two places) the side of a cubical block of cast iron weighing a ton, if iron weighs 7·2 as much as water, and a cubic foot of water weighs 1000 oz.

(206) A crown made of an alloy of copper and gold weighs 16·5 oz., while the water it displaces weighs $1\frac{1}{2}$ oz. How much copper does it contain, gold weighing 19·3 and copper 8·96 times as much as water?

(207) A grocer buys some tea at 4*s.* per lb. and some at 5*s.* 6*d.* In what proportion must he mix the two quantities so as to gain 20 % by selling the mixture at 6*s.* per lb.?

(208) Express ·583, $\cdot58\dot{3}$, $\cdot5\dot{8}\dot{3}$ and $\cdot\dot{5}8\dot{3}$ in the duodecimal scale to 5 places.

(209) I bought £333. 6s. 8d. 3 per cent. Consols for the benefit of an old servant, but wished to raise his income to £25 a-year by means of an investment in £50 mining shares, all paid up; the shares are at $8\frac{3}{4}$ premium, and the dividends are $7\frac{1}{2}$ % on the paid-up capital. How much mining stock must I buy, and what will it cost me?

(210) A Lithuanian league is 9769 yards long. Find the third convergent to the fraction that an English mile is of this.

(211) Express in inches the length of a French metre from the data that a metre is one ten-millionth of a quarter of the earth's circumference, and that the circumference is 3·14159 times the diameter 7911·7 miles.

(212) State:

a. Which vulgar fractions will yield recurring and which non-recurring decimal fractions.

b. If non-recurring, how the number of places can be foretold.

c. If recurring, whether the decimal will be mixed or pure.

(213) Find the value of a mass of silver weighing 15 lbs., 9 oz., 10 dwts., 20 grs., of which $\frac{14}{17}$ is pure, at the rate of $57\frac{7}{8}d.$ per oz. standard silver. (Standard silver contains $\frac{11\cdot 1}{12}$ pure silver.)

(214) If income-tax be 6d. in the £ and interest 5 %, how much do I gain or lose on an income of £1200 a-year by paying the whole year's tax at the end of the third quarter instead of paying it in 4 instalments at the end of each quarter?

(215) Express $\frac{3}{8}$ and $419\frac{1}{7}$ in the binary, ternary, quaternary, quinary, senary, septenary, octonary, nonary, decimal and duodecimal scales.

(216) Any two numbers whose units' figures are odd, but not 5, and whose difference is a power of 10, must be prime to one another. Prove this.

(217) If the wages of a woman are $\frac{4}{7}$ of the wages of a man, and it would require 8 men to earn a given sum of money, how many women must be added to 5 men to earn double the money? Explain your answer.

(218) Of the boys in a school, one-third are over 15 years of age, one-third between 10 and 15. A legacy of £100 can be exactly divided amongst them by giving 10*s.* to each boy over 15, 6*s.* 8*d.* to each between 10 and 15, and 3*s.* 4*d.* to each of the rest. How many boys are there in the school?

(219) Express the fraction $\frac{7}{25}$:

a. As a decimal fraction.

b. As a septenal fraction.

c. Also as a quaternal fraction.

(220) The mint price of gold is £3. 17*s.* $10\frac{1}{2}$*d.* per oz. standard. Find the smallest exact number of ounces that can be coined into an exact number of sovereigns.

(221) Find the weight of a cubical mass of iron whose edge is 2 ft., 5 in., 3 pts., if the iron is 7·157 times as heavy as water, and a cubic foot of water weighs 1000 oz. av.

(222) How high is a hill whose ascent is $1\frac{7}{8}$ miles in length, if the road rises $\frac{1}{\frac{3}{7}}$ inches in 55·25 feet?

(223) A can copy 6 pages while B copies 5, B copies 15 while C copies 12, and C can copy 4 while D copies 3; A who can write 20 pages a day receives a paper of 240 pages to copy; and after doing a quarter of it calls in B, C and D to help him. When will the work be finished?

(224) A merchant buys 1260 quarters of corn, $\frac{1}{5}$ of which he sells at a gain of 5 %, $\frac{1}{3}$ at a gain of 8 %, and the remainder at a gain of 12 %; if he had sold the whole at a gain of 10 % he would have gained £23. 2*s.* more. What was the cost price per quarter?

(225) The sum of £10552. 4*s.* 2*d.* is divided among 1000 persons in the ratio of the first thousand natural numbers. Find the share of the 150th person.

(226) The diameter of the fore wheel of a waggon is 3 ft., 6 in., that of the hind wheel 6 ft., 5 in. If two nails, one on the outside of each wheel, touch the ground together, in how many seconds (to two places) will they do so again, reckoning diameter : circumference = 1 : 3·14159, and the rate of travelling $4\frac{1}{2}$ miles an hour?

(227) I bought in London \$1000 American 5-20 bonds at 87 (i.e. 87 × 4*s.* 6*d.* per \$100 bond). What percentage shall I get for my money if the coupons in New York fetch clear of expense 4*s.* $0\frac{1}{2}d.$ per dollar, the bonds paying 6 % ?

(228) I bought in New York \$1500 5-20 bonds, brokerage $\frac{1}{8}$, at 111, in (paper) currency; gold was $15\frac{1}{2}$ premium; I paid by bill upon England, and the rate of exchange was 110 (i.e. \$110 in gold for 100 × 4*s.* 6*d.* payable in England); I re-sold these bonds in England at 89 (see last question), brokerage $\frac{1}{8}$. Find my profit or loss.

(229) The first term of an A. P. is $\frac{1}{4}$, the common difference is $\frac{1}{2}$. Find the 50th term.

(230) Suppose a debt can be discharged in a year by paying 1*s.* the 1st day, 2*s.* the second, and so on. What is the amount of the debt ?

(231) How many strokes do the clocks of Venice, which go on to 24 o'clock, strike in a day ?

(232) Find $(47\cdot\dot{3}18\dot{4})^6$ to 3 places.

(233) Find the equated time for the following amounts: £50 due in 6 months, £60 in 7 months, and £80 in 10 months, interest at 5 %, both by average and discount.

(234) A debt is to be paid as follows: $\frac{1}{4}$ at 2 months, $\frac{1}{8}$ at 3 months, $\frac{1}{8}$ at 4 months, $\frac{1}{8}$ at 5 months, and the balance at 6 months. What is the correct equated time to pay the whole, interest at 5 % ?

(235) An island is 73 miles in circumference, and 3 pedestrians all start together to travel round it in the same direction; the first goes 15, the second 17·5, and the third 10 miles a day. When will all three be again together ?

(236) A owes me the following sums: £480 due in $3\frac{1}{2}$ months, £607 due in 2 months, £577. 15*s.* due in 5 months. What sum should I accept as a single payment at the end of six months, reckoning interest at 4 % ?

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(237) What effect is produced on (*a*) the sum, (*b*) the difference, of two numbers, if the same quantity is added to each ?

(238) What effect is produced on (*a*) the sum, (*b*) the difference, of two numbers, if the same quantity is added to one and subtracted from the other number ?

(239) What effect is produced on (*a*) the product, (*b*) the quotient, of two numbers, if both numbers are multiplied by the same number?

(240) What effect is produced on the remainder, if (*a*) the divisor, (*b*) the dividend, be increased by a number not large enough to affect the quotient?

(241) What effect is produced on the remainder, if both divisor and dividend are (*a*) multiplied, (*b*) divided by the same number?

(242) State the conditions of increase in the value of a fraction, if the same number be added to both its terms.

(243) What effect is produced on the ratio, if the antecedent is multiplied and the consequent divided by the same number?

(244) What effect is produced on the square of a number, if the number is increased by a given number?

(245) Find the difference between the sum of the squares and the square of the sum of two numbers.

(246) What effect is produced on (*a*) the sum, (*b*) the difference, of two numbers, if each is multiplied by the same number?

(247) What effect is produced on (*a*) the L.C.M., (*b*) the G.C.M., (*c*) the average, of several numbers, if each is multiplied by the same number?

(248) How must a number be altered to double its reciprocal?

(249) To what limits do the *terms* of the two following series approach:

$$a.\ \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \ldots\ldots \qquad b.\ \tfrac{1}{7}, \tfrac{2}{8}, \tfrac{3}{9}, \tfrac{4}{10}, \ldots\ldots$$

and find the first term in each which differs from the limit by a quantity less than ·000001.

(250) To find the interest at 3 % per annum on any number of pounds for any number of days, multiply the number of pounds by twice the number of days, deduct $\frac{1}{70}$ of the product, and cut off the last two figures; the result will be the interest in pence. Shew that the error in the interest given by this rule for any time less than a year cannot exceed a shilling on every £2800 principal.

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C. 7.